NUCLEAR PHYSICS
(For Undergraduate and Postgraduate Students of
Indian Universities)

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Optical model
PREFACE TO THE SECOND EDITION

In this edition of the book, only minor changes have been made in some chapters. In the chapter on Nuclear Models (Ch. IX), the discussions on the individual particle model has been shortened to some extent and the relevant references have been added where the readers can get the details.

In the chapter on Elementary Particles (Ch. XVIII), the concept of the colour charge has been explained more elaborately. The introductory ideals of quantum chromodynamics have also been elaborated.

The author will feel happy if the book continues to enjoy the same popularity as before.

AUTHOR
PREFACE TO THE FIRST EDITION

It has been about two years since the first volume of Atomic and Nuclear Physics was published. The delay in the publication of the second volume is mainly due to the expansion in the scope of its coverage. This volume deals with the Nuclear Physics part of the subject taught in the postgraduate classes of the Indian Universities at a general level. However the special requirements of the advanced undergraduate (B.Sc. Honours students) have also been kept in mind. Accordingly the topics have been developed by starting from relatively elementary levels and subsequently going into more detailed discussions suitable for the postgraduate students.

The book grew out of the courses taught by the author in both undergraduate honours and postgraduate classes over the years. Basic knowledge of elementary quantum mechanics, special theory of relativity, statistical mechanics and matrices is assumed. The first three of these are included in Vol. 1 of the book.

The topics included are the basic requirements of students aspiring to pursue a career in physics. Though the main emphasis has been in the field of low energy nuclear physics, a brief discussion on elementary particle physics including high energy collisions has been included in view of the current trends.

As in Vol. 1 of the book, the basic principles of the concepts developed have been emphasized. A few problems are included at the end of the chapters, the working out of which should help achieve this objective.

A list of references is also included at the end of the different chapters which gives extensive coverage of the topics. References are also given to some of the classic papers dealing with the discoveries which are considered landmarks in the development of nuclear physics.

I am thankful to D. Roy Chowdhury for going through the chapter on Elementary Particle Physics and making many helpful suggestions which have been incorporated. My special thanks are due to Messers S. Chand & Company Ltd., specially to Sri Ravindra Kumar Gupta, Director of the Company, for his interest in bringing out this volume.

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S.N. Ghoshal

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1.1 Introduction

We have seen in Chapter I of the first volume* of this book that the atomic view of matter was first propounded by the British chemist John Dalton at the beginning of the nineteenth century. It took nearly a hundred years before the internal structure of the atom could be understood. J.J. Thomson's researches on the discharge of electricity through gases led to the discovery of the electron, which may be said to be the beginning of our understanding of the atomic structure. Thomson proposed an atomic model, discussed in Chapter 4 of Vol. I, in which the negatively charged point electrons were thought to be embedded within a body of positively charged matter, spread over the whole volume of the atom having a radius of the order of $10^{-10}$ m. This plum pudding model was however discarded in favour of the nuclear model of the atom proposed by Ernest Rutherford on the basis of the studies on the scattering of α-particles by matter.

1.2 Rutherford's theory of α-particle scattering

Alpha particles are highly energetic positively charged particles, emitted by radioactive substances, the amount of their positive electricity being equal to two units of electronic charge. Their mass is equal to that of a helium atom. They are emitted during the radioactive disintegration of certain heavy elements like uranium, or radium (see § 31). When the mono-energetic α-particles emitted from a radioactive substance are allowed to pass through a very thin metal foil, they are found to be scattered in different directions with respect to the direction of the collimated beam of the incident particles. Though, by far, a great majority of the particles are scattered at small angles, a small fraction is found to be scattered at large angles (greater than 90°).

Such large angle scattering cannot be explained on the basis of the Thomson model of the atom, the reason for which can be understood

* Throughout the present volume we shall refer to the first volume of this book simply as volume I.
qualitatively as follows. The electrostatic repulsive force between the positive charge of the scattering atom and the \( \alpha \)-particle depends inversely upon the square of the distance \( q \) of the \( \alpha \)-particle from the centre of the charge of the latter and directly upon the portion of the positive charge of the scattering atom contained within the sphere of radius \( q \) (see Fig. 1.1). As we shall see, the angle of scattering increases with the increasing effective charge of the atom, repelling the \( \alpha \)-particle and decreases with the increasing distance from the centre of the atom at which the \( \alpha \)-particle passes by the atom. When the \( \alpha \)-particle passes by at a relatively large distance from the centre of the atom (~ \( 10^{-10} \) m near the periphery of the atom), this distance becomes so large that the angle of scattering is quite small, even though the entire positive charge of the atom repels it.

On the other hand, when the \( \alpha \)-particle passes by at a relatively close distance from the centre of the atom, the effective charge repelling it is so small that the angle of scattering is again quite small.

To explain the observed large angle scattering of the \( \alpha \)-particles, Rutherford (1911) proposed a new model of the atom. According to him, the entire positive charge of an atom is contained within a very small sphere near the centre. This positively charged heavy core is known as the atomic nucleus. When an \( \alpha \)-particle passes by very close to the centre of the atom, it feels a strong electrostatic repulsive force due to the entire positive charge of the atom and hence is scattered at a relatively large angle.

During the scattering experiment, a narrow collimated beam of mono-energetic \( \alpha \)-particles is allowed to fall on a very thin foil of metal such as gold \( (Z = 79) \) or silver \( (Z = 47) \) which can be rolled or beaten to thicknesses of the order of \( 10^{-7} \) m (0.1 micron) or less. The \( \alpha \)-particles do not lose any appreciable energy in passing through such thin foils. Arrangements are provided to count the number of \( \alpha \)-particles scattered in different directions. In Fig. 1.2, a schematic diagram of the basic experimental arrangement for the study of \( \alpha \)-particle scattering is shown.

The force experienced by the \( \alpha \)-particle is a central force varying inversely as the square of its distance from the nucleus. It is known from the laws of mechanics that the trajectory of a particle acted upon by such a force is a conic.

The conic will be an ellipse, a parabola or a hyperbola, depending on whether the initial energy of the particle \( E < 0 \), \( E = 0 \) or \( E > 0 \). In the present case the initial energy of the \( \alpha \)-particle is positive \( (E_\alpha > 0) \).

So its trajectory \( \text{PAP}' \) is a hyperbola with the scattering nucleus located at one of its foci \( F \). Here we assume that the entire positive charge \( +Ze \) of the nucleus is concentrated at the point \( F \).

In Fig. 1.3, the trajectory \( \text{PA}' \) of an \( \alpha \)-particle under the action of the electrostatic repulsion due to a nucleus of charge \( +Ze \) is shown.

Let \( M \) and \( Ze \) be the mass and charge of the \( \alpha \)-particle; here \( Z = 2 \). The nucleus is assumed to be infinitely heavy.

As the \( \alpha \)-particle approaches the nucleus \( F \) from a great distance, the electrostatic repulsion on it is negligibly small when it is very far away. In this case its potential energy is negligible and the entire energy is kinetic: \( E_\alpha = \frac{1}{2} Mv^2 \) where \( v \) is the initial velocity of the \( \alpha \)-particle. The path of the \( \alpha \)-particle at a great distance is thus a straight line, coinciding with the asymptote \( QOR \) of the hyperbola \( \text{PA}' \). If we drop the perpendicular \( FG \) from the nucleus to this asymptote line, then its length \( FG = b \) is known as the impact parameter.

As the \( \alpha \)-particle approaches the nucleus, its trajectory bends more and more away from \( F \) due to the increasing electrostatic repulsion. Finally, when it reaches the point \( A' \) at a minimum distance from \( F \), it begins to move away from the nucleus along \( A'F' \). At a great distance from the nucleus, its path of recession coincides with the asymptote \( R'OQ' \) of the hyperbola. The angle \( R'OQ' \) between the two asymptotes \( QOR \) and \( R'OQ' \) is the angle of scattering \( \theta \). The two asymptotes are equally inclined at an angle \( \phi \) to the axis \( FF' \) of the hyperbola.

When the \( \alpha \)-particle reaches the vertex \( A' \) of the hyperbola, the electrostatic repulsion on it is the maximum. Writing the least distance of approach \( FA' = q \), the potential energy of the \( \alpha \)-particle at \( A' \) is

\[ \text{potential energy} = qZeU = qZe \cdot \frac{Ze}{r^2} = \frac{q^2Z^2}{r^2}, \]

where \( U \) is the potential energy of the force at \( r \) and \( r \) is the distance from the nucleus.

\[ \tan \theta = \frac{b}{d} = \frac{b}{c + d} = \frac{Ze}{Mv^2}, \]

where \( b \) is the impact parameter and \( c + d \) is the distance from the nucleus.

\[ \phi = \tan^{-1} \left( \frac{Ze}{Mv^2} \right), \]

where \( \phi \) is the angle of scattering.
\[ V = \frac{2e^2}{4\pi \varepsilon_0} q \] where \( \varepsilon_0 = 10^{-9}/36\pi \text{ F/m} \) is the dielectric constant of empty space.

The velocity \( v_m \) of the \( \alpha \)-particle at \( A' \) is the minimum. Its kinetic energy at this point is \( E_k = \frac{1}{2} M v_m^2 \) so that the total energy is

\[ E = E_k + V = \frac{1}{2} M v_m^2 + \frac{2e^2}{4\pi \varepsilon_0} q \]

Equating this to the initial energy of the \( \alpha \)-particle we get

\[ E = \frac{1}{2} M v_m^2 + \frac{2e^2}{4\pi \varepsilon_0} q \]

or,

\[ \frac{v_m^2}{v^2} = 1 - \frac{2e^2}{2\pi \varepsilon_0 M v^2 q} \]

Again the linear momentum of the \( \alpha \)-particle at a great distance is \( Mv \) along the line \( QOR \) and hence its initial angular momentum is

\[ Mv. FG = Mv. b \]

At \( A' \), the linear momentum \( Mv_m \) of the \( \alpha \)-particle is perpendicular to the axis \( FF' \) of the hyperbola, so that its angular momentum at this point is

\[ Mv_m. FA' = Mv_m q \]

Since the force acting on the \( \alpha \)-particle is central, its angular momentum must be conserved, according to the well-known laws of mechanics. So we have

\[ Mv b = Mv_m q \]

Referring to Fig. 1.3, we know from the properties of the hyperbola that

\[ OF = OF' = \varepsilon \cdot OA' \]

where \( \varepsilon > 1 \) is the eccentricity. So

\[ q = FA' = OF + OA' = OF \left( 1 + \frac{1}{\varepsilon} \right) \]

Again from the theory of conics, we know that

\[ \frac{1}{\varepsilon} = \cos \phi \]

so that

\[ q = OF (1 + \cos \phi) \]

Since \( \sin \phi = FG/OF \), we have

\[ q = FG (1 + \cos \phi)/\sin \phi = b (1 + \cos \phi)/\sin \phi \]

From Eqs. (1.2-3) and (1.2-4), we get

\[ \frac{b}{v_m} = \frac{\sin \phi}{v} \quad \frac{1}{\cos \phi} \]

Then from Eqs. (1.2-2) and (1.2-5), we have

\[ 1 - \frac{2e^2}{2\pi \varepsilon_0 M v^2 q} = \frac{\sin^2 \phi}{1 + \cos \phi} = \frac{1 - \cos \phi}{1 + \cos \phi} \]

or,

\[ \frac{2e^2}{2\pi \varepsilon_0 M v^2 q} \sin \phi = 2 \cos \phi \]

Again since \( \phi + \frac{\theta}{2} = \frac{\pi}{2} \), we get

\[ b = \frac{2e^2}{4\pi \varepsilon_0 M v^2} \cot \frac{\theta}{2} \]

Eq. (1.2-7) shows that as the impact parameter \( b \) becomes smaller, the angle of scattering \( \theta \) increases. In Fig. 1.4, this dependence of \( \theta \) on \( b \) is shown schematically.

During the scattering experiment, a very large number of \( \alpha \)-particles is projected towards the target nuclei. If we confine our attention on a single scattering nucleus, then the \( \alpha \)-particles approaching it along different asymptotic paths will have different impact parameter \( b \). If we draw two infinite coaxial cylinders of radii \( b \) and \( (b + db) \) having their axis parallel to the incident direction with the nucleus on the axis, then all \( \alpha \)-particles with their asymptotic paths between these two cylinders will be scattered between the two angles \( \theta \) and \( \theta - d \theta \) determined by Eq. (1.2-7) (see Fig. 1.5). The annulus between the projections of these two cylinders on a plane perpendicular to the axis has obviously an area \( \pi (b + db)^2 \) so that the number of \( \alpha \)-particles incident on this annulus in a given interval of time is

\[ dN = N \pi b^2 \text{ d} \theta \]

where \( N \) is the number incident per unit area. Then using Eq. (1.2-7), we get the number scattered between \( \theta \) and \( \theta - d \theta \) as

\[ dN = -\pi N \left( \frac{2e^2}{4\pi \varepsilon_0 M v^2} \right)^2 \cot^3 \frac{\theta}{2} \cos \frac{\theta}{2} d \theta \]

All these \( \alpha \)-particles are scattered into the solid angle...
Thus the number scattered per unit solid angle is (writing \( Z = 2 \))

\[
\frac{dN}{d\Omega} = N\left(\frac{Ze^2}{4\pi\varepsilon_0 M v^2}\right) \csc^2 \frac{\theta}{2}
\]

In Eq. (1.2-9), the negative sign appearing in Eq. (1.2-8) has been dropped. This negative sign simply indicates that as the impact parameter \( b \) increases, the scattering angle \( \theta \) decreases.

In the above deduction, it is assumed that out of \( N \) \( \alpha \)-particles incident on a unit area of the scatterer, the number \( dN \) is scattered by the action of a single scattering nucleus. So if only one \( \alpha \)-particle is incident per unit area, so that \( N = 1 \), then Eq. (1.2-9) gives the probability of scattering of that particle due to the action of a single scattering nucleus. The value of this probability of scattering per unit solid angle is thus given by

\[
\frac{d\sigma}{d\Omega} = \left(\frac{Ze^2}{4\pi\varepsilon_0 M v^2}\right) \csc^2 \frac{\theta}{2}
\]

Eq. (1.2-10) is known as the Rutherford scattering formula.

Now \( Ze^2/4\pi\varepsilon_0 \) has the dimensions of \([\text{energy} \times \text{length}]\) while \( Mv^2 \) has the dimensions of energy. Hence \( (Ze^2/4\pi\varepsilon_0 M v^2) \) has the dimensions of length. Hence \( d\sigma/d\Omega \) has the dimension of the square of length, i.e., of an area. For this reason, the probability of scattering given by Eq. (1.2-10) is designated as the cross section of scattering per unit solid angle or differential scattering cross section. Since the element of solid angle \( d\Omega \) is dimensionless, \( d\sigma \) has the dimension of an area and measures the cross-section of scattering into the solid angle \( d\Omega \). The total scattering cross section \( \sigma \) obtained by integrating \( d\sigma \) over all solid angles has also the dimension of an area.

In the scattering experiment, a collimated beam of mono-energetic \( \alpha \)-particles is allowed to fall on a scattering foil of thickness \( t \) as shown in Fig. 1.6. If \( N \) \( \alpha \)-particles are incident on the foil in a given interval of time the number \( \Delta N \) which is scattered into the solid angle \( \Delta \Omega \) at the angle \( \theta \) will be proportional to \( N \Delta \Omega \). It will also depend on the number of scattering nuclei \( n_i \) present per unit area of the foil (i.e., the density of packing of the nuclei) so that we can write

\[
\Delta N \propto N n_i \Delta \Omega
\]

\[
\frac{\Delta N}{d\Omega} = \frac{d\sigma}{d\Omega} \propto N n_i \Delta \Omega = N n_i \Delta \sigma
\]

From Fig. 1.6, we see that \( n_i = n t \) where \( n \) is the number of scattering nuclei per unit volume of the foil. So we get, using Eq. (1.2-10)

\[
\Delta N = N n t \left(\frac{Ze^2}{4\pi\varepsilon_0 M v^2}\right) \csc^2 \frac{\theta}{2} \Delta \Omega
\]

From Fig. 1.7, we see that if we draw a sphere of radius \( r \) and draw two cones of semi-vertical angles \( \theta \) and \( \theta + \Delta \theta \), then the area of the strip between the two cones on the surface of the sphere is

\[
\Delta S = 2\pi r^2 \sin \theta \Delta \theta = 2\pi r \sin \theta \Delta \theta = r^2 \Delta \Omega
\]

The \( \alpha \)-particles scattered into the solid angle \( \Delta \Omega \) at the scattering angle \( \theta \) pass through this strip. Hence the number of scattered \( \alpha \)-particles falling on a unit area of this strip is

\[
N_s = \frac{\Delta N}{\Delta \Omega} = N n t \left(\frac{Ze^2}{4\pi\varepsilon_0 M v^2}\right) \csc^2 \frac{\theta}{2}
\]

1.3 Experimental verification of Rutherford's scattering formula by Geiger and Marsden

From Eq. (1.2-12) we see that

\[
(a) \quad N_s \propto \frac{1}{\sin^4 \theta}
\]

\[
(b) \quad N_s \propto t
\]

\[
(c) \quad N_s \propto \frac{1}{E^2} \quad (E = \frac{1}{2} M v^2 \text{ is the energy of the } \alpha \text{-particles})
\]

\[
(d) \quad N_s \propto Z^2
\]

The above conclusions of Rutherford's theory were tested by a series of experiments performed by H. Geiger and E. Marsden (1913), two of Rutherford's associates.
Their experimental arrangement is shown in Fig. 1.8. R is a very thin metal foil (of gold or silver). R is a thin walled glass tube enclosing a small amount of radioactive radon gas ($^{226}$Rn) emitting mono-energetic $\alpha$-particles, which after collimation by an opening through the shield A fall on the scattering foil F. After scattering from F through different angles $\theta$ they fall on the fluorescent screen S to produce scintillations (see Ch VII ) which can be observed through the microscope M that can be turned around the axis. The whole apparatus is kept under high vacuum by means of a vacuum pump. By counting the number of scintillations produced on S, the number $N_s$ of $\alpha$-particles incident on unit area of it can be found.

Geiger and Marsden found that when $\alpha$-particles of a definite velocity $v$ are scattered from a foil of given thickness $t$, at different angles $\theta$, the product $(N_s \times \sin^4 \theta / 2)$ remained constant within the limits of experimental error, as can be seen from the last column of Table 1.1. This provided a confirmation of the conclusion (a) above.

<table>
<thead>
<tr>
<th>Angle of scattering</th>
<th>$\sin^4 \theta / 2$</th>
<th>No. of $\alpha$-particles falling on a unit area of the scintillating screen ($N_s$)</th>
<th>$N_s \times \sin^4 \theta / 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>15°</td>
<td>2.903 x 10^{-4}</td>
<td>132,000</td>
<td>38.4</td>
</tr>
<tr>
<td>30°</td>
<td>4.484 x 10^{-3}</td>
<td>7,800</td>
<td>35.0</td>
</tr>
<tr>
<td>45°</td>
<td>2.146 x 10^{-2}</td>
<td>1,435</td>
<td>30.8</td>
</tr>
<tr>
<td>60°</td>
<td>0.0625</td>
<td>477</td>
<td>29.8</td>
</tr>
<tr>
<td>75°</td>
<td>0.1379</td>
<td>211</td>
<td>29.1</td>
</tr>
<tr>
<td>90°</td>
<td>0.3952</td>
<td>69.5</td>
<td>27.5</td>
</tr>
<tr>
<td>120°</td>
<td>0.5586</td>
<td>51.9</td>
<td>29.0</td>
</tr>
<tr>
<td>135°</td>
<td>0.7245</td>
<td>43.0</td>
<td>31.2</td>
</tr>
<tr>
<td>150°</td>
<td>0.8695</td>
<td>33.1</td>
<td>28.8</td>
</tr>
</tbody>
</table>

Geiger and Marsden next measured the number of $\alpha$-particles scattered through a given angle ($\theta = \text{constant}$) using scattering foils of different thickness. Their results are shown graphically in Fig. 1.9 in which the number scattered $N_s$ into a unit area is plotted as a function of the foil thickness. The thickness is expressed in terms of the air-equivalent, which is the thickness of air that would produce the same

Geiger and Marsden found that when $\alpha$-particles of a definite velocity $v$ are scattered from a foil of given thickness $t$, at different angles $\theta$, the product $(N_s \times \sin^4 \theta / 2)$ remained constant within the limits of experimental error, as can be seen from the last column of Table 1.1. This provided a confirmation of the conclusion (a) above.

| Range of the incident $\alpha$-particles (cm) | $1/t$ (relative values) | No. of $\alpha$-particles falling on a unit area of the scintillating screen ($N_s$) | $N_s \times v^4$ |
|--------------------------------------------|-------------------------|--------------------------------------------------------------------------------|-----------------
| 5.5                                       | 1.0                     | 24.7                                                                           | 25              |
| 4.76                                      | 1.21                    | 29.0                                                                           | 24              |
| 4.05                                      | 1.50                    | 33.4                                                                           | 22              |
| 3.32                                      | 1.96                    | 44.0                                                                           | 22              |
| 2.51                                      | 2.84                    | 81.0                                                                           | 28              |
| 1.84                                      | 4.32                    | 101.0                                                                          | 28              |
| 1.04                                      | 9.22                    | 255.0                                                                          | 28              |

To test the conclusion (c), Geiger and Marsden used $\alpha$-particles of different velocities. If a collimated beam of mono-energetic $\alpha$-particles of given initial energy is allowed to pass through a thin mica foil, then the particles emerging from the foil lose a definite amount of energy, depending on the thickness of the foil. The emergent $\alpha$-particles thus constitute an approximately mono-energetic beam of reduced energy. If these are scattered from a scattering foil, then the scattering of the $\alpha$-particles at a reduced velocity can be studied. This was done by Geiger and Marsden using mica foils of different thicknesses to reduce the energy to different values and then studying their scattering at a
particular angle ($\theta = \text{constant}$) by a scattering foil of given thickness ($t = \text{constant}$). The velocity $v$ of the $\alpha$-particles could be determined by measuring their range $R$ (see § 4.7). They found that the product of $(N_q \times v^2)$ was a constant, as can be seen from the last column of Table 1.2. In this way the law of velocities (conclusion $c$) of Rutherford's theory was verified.

Because of the large errors in Geiger and Marsden's measurements, the last conclusion ($d$) could not be verified by them, which however was done later (see § 1.5).

1.4 Rutherford's nuclear model of the atom

Experimental verification of Rutherford's theory of the scattering of $\alpha$-particles by Geiger and Marsden, discussed in the previous section, solidly established the nuclear model of the atom proposed by Rutherford. According to this model, the entire positive charge and almost the entire mass of the atom are concentrated within a sphere of radius $R$, small compared to the atomic radius, which is of the order of $10^{-10}$ m (see Chapter I, Vol. I).

Rutherford's theory of the scattering of the $\alpha$-particles gives us some idea of the size of the sphere within which the nuclear charge is contained. Eqs. (1.2-5) and (1.2-6) show that the minimum distance of approach $q$ of the $\alpha$-particle to the nucleus depends on the angle of scattering $\theta$ and decreases with increasing $\theta$. When $\theta = 180^\circ$, the value of the least distance of approach $q_m$ is such that the $\alpha$-particle turns back along the path of incidence from the point of closest approach (turning point) and hence comes to a momentary rest ($v = 0$) at this point. So its energy is wholly potential at this point and may be written as

$$E = \frac{1}{2}Mv^2 = \frac{Ze^2}{4\pi\varepsilon_0 q_m}$$

which gives

$$q_m = \frac{2Ze^2}{2\pi\varepsilon_0 E} = \frac{Ze^2}{4\pi\varepsilon_0 E} \cdots (1.4-1)$$

For $\alpha$-particles from $^{222}$Rn isotope having the energy $E_\alpha = 5.486$ Mev we get $q_m = 2.47 \times 10^{-14}$ m for silver ($Z = 47$) and $q_m = 4.15 \times 10^{-14}$ m for gold ($Z = 79$). Since the radius of the atomic nucleus must be smaller than the above estimates, it is obvious that the nuclear radii are smaller by factors of the order of $10^4$ to $10^5$ compared to the atomic radii.

In the above calculations of $q_m$, it has been assumed that Coulomb's law of electrostatic interaction between the $\alpha$-particles and the scattering nucleus holds good even at such short distance.

Geiger and Marsden's experiments provide conclusive evidence in favour of this. In their experiments, the largest angle of scattering was $150^\circ$ and not $180^\circ$. In this case the minimum distance of approach is slightly higher than $q_m$ as estimated above. Nevertheless, the above discussions show definitely that Coulomb's law of force between the scattering nucleus and the $\alpha$-particle holds at distances of the order of $10^{-14}$ m.

Rutherford scattering of alpha particles and...

Since the atom as a whole is electrically neutral, there must be a negatively charged counterpart of the atom, in which the total negative charge must be equal to the total positive charge of the nucleus. As we have seen in § 4.3, Vol. I, this negatively charged part of the atom actually comprises the negatively charged electrons which rotate in a number of specific orbits of radii comparable to the atomic radii ($\sim 10^{-10}$ m) like the planets revolving round the sun. This picture of the atom originally due to Rutherford, is fundamentally different from the plum-pudding model of the atom due to J.J. Thomson.

Comparison of the nuclear radius ($\sim 10^{-15}$ m) with the atomic radius ($10^{-10}$ m) shows that the former is about $10^5$ times smaller than the latter. Further, the strong binding of the constituents of the nuclei results from forces which must have very short range of action, less than at least $2 \times 10^{-15}$ m.

However, Rutherford model of the atom has one serious drawback. Such an atom cannot be a stable configuration. The electromagnetic theory of light predicts that the revolving electrons, due to their centripetal acceleration, should continually emit e.m. radiation so that they would move spirally inwards and ultimately plunge into the nucleus. It was left to Niels Bohr of Denmark (1913) to suggest a way out of the difficulty, which however involved entirely new concepts and that were at variance with some of the fundamental concepts of classical mechanics and of Maxwell's electromagnetic theory of light.

This is known as Bohr's quantum theory. The quantum theory, in a more developed form at present, constitutes the theoretical basis of all subatomic phenomena. This new theory, known as quantum mechanics, has been briefly discussed in Vol. I of this book.

1.5 Chadwick's determination of nuclear charge.

James Chadwick, a student of Rutherford, using improved techniques, was the first (1920) to determine the nuclear charge of several elements on the basis of Rutherford's theory of $\alpha$-particle scattering. We have seen that the number of $\alpha$-particles scattered by a nucleus of charge $4Z$ is proportional to the square of the nuclear charge; i.e. $N_\alpha \propto Z^2$ (conclusion $d$ in § 1.3).

Chadwick's apparatus is shown in Fig. 1.10. To increase the number of the scattered $\alpha$-particles, Chadwick used a narrow ring-shaped scattering foil RR' mounted on a suitable frame, as shown in Fig. 1.10(a).

S is the $\alpha$-source and D is a screen of some scintillating material used as the detector. The ring RR' was so placed that its plane was perpendicular to the line SD which passed through the centre of the ring.

The ring RR' was thus enclosed between two coaxial cones with S at the vertex. S and D were equidistant from the plane of RR' so that SR = RD. If the semi-vertical angle of the inner cone is $\theta/2$ as shown in Fig. 1.10(a), then both SR and RD are inclined to the axis SD at $\theta/2$. If an $\alpha$-particle is incident on the ring RR' along SR and is then scattered along KD to reach the detector, the mean angle of scattering will be $\theta$. This is...
true for all points on RR'. Thus even though the width of the scattering foil RR' was quite small in Chadwick's experiment the effective area presented by it to the incident α-particle for scattering at a given angle was relatively large, which helped increase the number of the scattered particles, thereby improving the statistics of counting.

A lead screen L prevented the α-particles from falling directly on the detector from the source. With L removed, it was possible to determine the number \( N_0 \) of α-particles falling directly on D per second. Knowing the solid angle subtended by D at S, it was thus possible to estimate the strength of the source, i.e., the rate of emission of the α-particles from it. If \( N_0 \) represents the source strength, A the area of D and SD = R, then the number falling on D is

\[
N_D = \frac{A}{4\pi R^2} N_0
\]

Knowing \( N_0 \) it was possible to determine the number \( N \) of α-particles falling on RR' per second. The solid angle subtended by RR' at S is

\[
\Delta \Omega = 2\pi \sin \frac{\theta}{2} \Delta (\frac{\theta}{2}) = \pi \sin \frac{\theta}{2} \Delta \theta
\]

\[
N = \frac{N_0}{4\pi \Delta \Omega} = \frac{N_0}{4} \sin \frac{\theta}{2} \Delta \theta
\]

and

\[
N = \frac{\pi R^2}{A} \sin \frac{\theta}{2} \Delta \theta
\]

... (1.5-1)

\( \Delta \theta \) could be calculated from the width of the ring so that \( N \) could be found with the help of Eq. (1.5-1).

To determine the nuclear charge \( Z_e \) with the help of Eq. (1.2-12) it is necessary to know the number of scattered α-particles falling on the detector per second. It should be noted that the α-particles scattered from RR', are incident on D making an angle \( \theta/2 \) with its normal. The projection of its surface area \( A \) normal to the path RD of the scattered particles is \( A \cos (\theta/2) \) [see Fig. 1.10(b)]. Hence to get the number of α-particles falling on a unit area of D, \( N_S \) given in Eq. (1.2-12) should be multiplied by \( \cos (\theta/2) \).

Further, since the α-particles were incident on the scattering foil RR' at an angle \( \theta/2 \) to its normal, the effective thickness of the foil presented to the α-beam was \( t/\cos (\theta/2) \) [see Fig. 1.10(c)]. So we get finally from Eq. (1.2-12) with the help of Eq. (1.5-1)

\[
N_S' = \frac{N_0}{\pi} \left( \frac{Z_e^2}{4\pi \epsilon_0 M v^2} \right)^2 \frac{\Delta \theta}{\cos (\theta/2)} \cos (\theta/2) \times \frac{t}{\cos (\theta/2)}
\]

Here \( r \) is the mean distance of the scattering foil RR' from S. If \( a \) is the mean radius of the ring RR', then \( a = r \sin \theta/2 \) so that \( r = a \csc \theta/2 \).

If \( w \) be the width of the scattering foil, then from Fig. 1.10a we see that

\[
\Delta \theta = w \cos \frac{\theta}{2} / a \csc \frac{\theta}{2}
\]

... (1.5-2)

We then get finally

\[
N_S' = \frac{2\pi N_0 R^2}{A} \left( \frac{Z_e^2}{4\pi \epsilon_0 M v^2} \right)^2 \cos \frac{\theta}{2}
\]

... (1.5-3)

Thus the number of scattered α-particles falling on a unit area of the detector is proportional to \( \cos \theta/2 \).

Chadwick determined \( Z \) for a number of metallic elements using the above formula. His results for copper, silver and gold are given below and are compared with their atomic numbers.

<table>
<thead>
<tr>
<th>Element</th>
<th>( Z ) as determined by Chadwick</th>
<th>Atomic Number</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cu</td>
<td>29.3</td>
<td>29</td>
</tr>
<tr>
<td>Ag</td>
<td>46.3</td>
<td>47</td>
</tr>
<tr>
<td>Au</td>
<td>77.4</td>
<td>78</td>
</tr>
</tbody>
</table>
It should be noted that the atomic number of an element represents the serial number of its position in the Periodic Table (see § 6.6, Vol. I) and is equal to the number of orbital electrons in the atom. The latter were determined directly by Barkla from x-ray scattering experiments and were found to be in agreement with the values of the atomic number determined from the study of the characteristic x-ray spectra by Moseley (see § 8.9, Vol. I). Since the atom as a whole is electrically neutral, the nuclear charge \( Z \) (measured in electronic charge unit) should be equal to the atomic number. Chadwick’s determination of \( Z \) for the different elements confirmed this most conclusively. The slight departures of these values of \( Z \) from the integral numbers were due to experimental error.

Chadwick’s experiment described above also established the validity of conclusion \( d \) of Rutherford’s scattering theory (see § 1.3) which, as we have seen, could not be verified by Geiger and Marsden.

References

Problems
1. Calculate the number of gold nuclei per cubic metre. \( A = 197 \) and density = 19.32 \( \times 10^3 \) kg/m\(^3\) for gold. Avogadro number \( N_0 = 6.022 \times 10^{23} \) molecules per mole. \( (5.9 \times 10^{27}) \)
2. A 6 MeV \( \alpha \)-particle is scattered by the nucleus of a mercury atom \( (Z = 80) \) at 120°. What is the minimum distance to which the \( \alpha \)-particle approaches the nucleus? What is the velocity of the \( \alpha \)-particle at this point? \( (4.14 \times 10^{-14} \text{ m}; 1.7 \times 10^7 \text{ m/s}) \)
3. Calculate the angular momentum quantum number of the \( \alpha \)-particle in the unit of \( h \) in Prob. 2.
4. The following data were obtained in an \( \alpha \) scattering experiment from silver \( (Z = 47) \). Check whether the data conforms to the theory of Rutherford scattering.

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\theta & 15° & 30° & 45° & 60° & 75° & 120° \\
\hline
n_\theta & 8.26 \times 10^4 & 5.09 \times 10^3 & 9.7 \times 10^2 & 3.2 \times 10^2 & 1.36 \times 10^2 & 33 \\
\hline
\end{array}
\]

Nuclear Structure and General Properties of Nuclei

2.1 Introduction
After having established the nuclear model of the atom, we are now in a position to discuss the general characteristics of the atomic nuclei in their ground states.

The nucleus of an atom, like the atom itself, is a bound quantum system and hence can exist in different quantum states characterized by their energies, angular momenta etc. The lowest energy state is known as the ground state and the nuclei normally exist in this state. The properties of the nuclei which will be discussed in this chapter correspond to their ground states and are usually called their static properties in contrast to the dynamic characteristics of the nuclei which are exhibited in the processes of nuclear reactions, nuclear excitation and nuclear decay. The important static properties of the nuclei include their electric charge, mass, binding energy, size, shape, angular momentum, magnetic dipole moment, electric quadrupole moment, statistics, parity and iso-spin. We shall discuss these properties in the next few sections. The experimental methods for their determination will be discussed in Ch. VIII.

2.2 Nuclear mass and binding energy
As we shall see later, the atomic nuclei are made up of two different types of elementary particles, protons and neutrons. The proton is nothing but the nucleus of the hydrogen atom, i.e., a hydrogen atom from which the single orbital electron has been removed. It carries one electronic unit of positive charge \( (+e) \) and has a mass about 1836 times the electronic mass \( (m_e) \). The neutron, on the other hand, is electrically neutral and is slightly heavier than the proton (see Table 2.1). The protons and the neutrons are held together inside the nucleus by very strong short range attractive force. This force is different from the more commonly known forces like the gravitational or electrical forces and constitutes what may be called the specifically nuclear interaction. The neutrons and the protons are jointly known as the nucleons.

The sum of the numbers of neutrons \( (N) \) and protons \( (Z) \) inside the nucleus is known as its mass number \( A \), so that \( A = N + Z \). Obviously \( A \) is an integer just as \( N \) and \( Z \) are. \( Z \) is equal to the atomic number of the element in the periodic table.
A nucleus of an atom X of atomic number Z and mass number A is symbolically written as \( ^A_X\text{X} \). For example, \(^2\text{He}\) denotes the nucleus of the helium atom of atomic number 2 and mass number 4. This is actually the \( ^6\text{He} \) particle. The proton and the neutron are symbolically written as \( p \) and \( n \). The subscript to the left of the atomic symbol denoting the atomic number is often omitted, since the chemical symbol determines uniquely the atomic number of the element, so that we can write \( ^2\text{H} \), \(^2\text{He}\), \(^2\text{Li}\) etc.

Nuclei with the same \( Z \), but different \( A \) are called isotopes. A particular element with a given \( Z \) may have isotopes of different mass numbers. Their nuclei contain the same number of protons, but different numbers of neutrons. As an example, the element lithium (\( Z = 3 \)) has two stable isotopes, \(^6\text{Li} \) and \(^7\text{Li} \). The number of protons in the nuclei of both is the same, viz., \( Z = 3 \). In \(^6\text{Li} \), the neutron number \( N = 3 \) while in \(^7\text{Li} \), \( N = 4 \).

Isotopes were first discovered amongst naturally radioactive elements. J. J. Thomson while exploring the properties of positive rays by the parabola method, was the first to discover stable isotopes of neon (\( Z = 10 \), viz., \(^{20}\text{Ne} \) and \(^{22}\text{Ne} \) (see § 2.8, Vol I)). Many elements are at present known to have more than one stable isotope, though some have only one stable isotope. Thus sodium (\( Z = 11 \)) has a single stable isotope viz., \(^{23}\text{Na} \) with \( N = 12 \). Elements having more than one stable isotope in the natural state are mixtures of these isotopes in fixed proportions, known as their relative abundances, which may vary more or less the same, irrespective of the source from which they are derived. Thus natural lithium is a mixture of the two stable isotopes \(^6\text{Li} \) and \(^7\text{Li} \) with the relative abundances 7.4\% and 92.6\% respectively.

Hydrogen has two stable isotopes \(^1\text{H} \) (99.99\%), \(^2\text{H} \) (0.01\%). The isotope \(^3\text{H} \) of hydrogen is called the deuterium and its nucleus deuteron. Another unstable isotope of hydrogen with \( A = 3 \) is called the tritium and it is known. Its symbol is \(^3\text{H} \).

In the table in Appendix A–VIII, the isotopes of all known naturally occurring elements with their relative abundances and atomic masses are listed.

Nuclei with the same \( A \), but different \( Z \) are known as isobars, while nuclei with the same number of neutrons are known as isotones.

Though in nuclear physics we are concerned with the masses of the nuclei, experimental determination using mass spectroscopes yield the atomic masses. Hence in all tables, it is the atomic masses which are shown and not the nuclear masses. The nuclear mass \( M_{\text{nuc}} \) is obtained from the atomic mass \( M \) by subtracting the masses of the \( Z \) orbital electrons from the latter.

\[
M_{\text{nuc}} = M - Z m_e
\]  

This expression is not exact, since it does not take into account the binding energies of the electrons in the atom. However, the error due to this is very small and hence in all numerical calculations involving the nuclear processes, it is the atomic masses which are used, since the electronic masses usually cancel out (see below).

As stated, above, the nuclei of the atoms are very strongly bound. It requires energies of the order of a few million electron volts (MeV) to break away a nucleon from a nucleus, compared to only a few electron-volts energy to detach an orbital electron from an atom to ionize it. (In the case of the hydrogen atom, this ionization energy is 13.6 eV; see § 4.6, Vol. I).

If we want to break up a nucleus of \( Z \) protons and \( N \) neutrons completely so that they are all separated from one another, a certain minimum amount of energy is to be supplied to the nucleus. This energy is known as the binding energy of the nucleus. Conversely if we start with \( Z \) protons and \( N \) neutrons at rest, all completely separated from one another, and then bring these together to constitute the nucleus of mass number \( A = N + Z \) and nuclear charge \( Z \), then an amount of energy equal to the binding energy of the nucleus will be evolved. Let us now see what the source of this energy is.

According to the Special Theory of Relativity propounded by Albert Einstein, mass and energy are equivalent (see § 15.3, Vol I). The mass of a body can be transformed into energy in certain physical and chemical processes and vice-versa. The mass \( m \) of a body, if completely converted into energy, produces an equivalent amount of energy \( \text{energy} = mc^2 \) where \( c \) is the velocity of light in vacuum: \( c = 2.997925 \times 10^8 \) m/s.

Thus 1 g, when completely converted into energy, gives \( 9 \times 10^{13} \) joules of energy.

In the case of formation of a nucleus the evolution of energy equal to the binding energy of the nucleus takes place due to the disappearance of a fraction of the total mass of the \( Z \) protons and \( N \) neutrons, out which the nucleus is formed. If the quantity of mass disappearing is \( \Delta M \), then the binding energy is

\[
E_b = \Delta Mc^2 \quad \ldots(2.2-1)
\]

From the above discussion, it is clear that the mass of the nucleus must be less than the sum of the masses of the constituent neutrons and protons. Denoting the masses of the hydrogen atom and the neutron as \( M_H \) and \( M_n \), we can then write

\[
\Delta M = ZM_H + NM_n - M(A, Z) \quad \ldots(2.2-2)
\]

where \( M(A, Z) \) is the mass of the atom of mass number \( A \) and atomic number \( Z \). Hence the binding energy of the nucleus is

\[
E_b = (ZM_H + NM_n - M(A, Z))c^2 \quad \ldots(2.2-3)
\]

In Eq. (2.2-3) the masses of the \( Z \) electrons cancel out on the right hand side and hence \( \Delta M \) for the nucleus is actually equal to the sum of

\[\text{binding energy}\]
the masses of \( Z \) protons \((ZM_p)\) and \( N \) neutrons \((NM_n)\) minus the nuclear mass \( M_{nuc} (A, Z) \) of the atom.

It may be noted that because of the mass-energy equivalence, the masses of the atoms can be expressed in energy units. If this is done, then in Eq. (2.2-3), \( c^2 \) on the r.h.s. may be omitted.

Various methods for the determination of the atomic masses very accurately have been developed. These will be discussed in detail in Ch. VIII.

The unit of atomic mass is defined to be one-twelfth of the mass of the atom \(^{12}\)C taken to be exactly 12 units and is designated by the symbol \( 'u' \) (unified atomic mass unit). This unit of atomic mass has been in use since 1961 by both physicists and chemists by international agreement. Prior to 1961, the atomic mass units used by physicists and chemists were different. The physicists' unit was previously taken to be one-sixteenth of the mass of \(^{16}\)O isotope (taken to be exactly 16 units) and was called the atomic mass unit (amu). The conversion factor from one scale to the other is given by

\[
1 \text{u} : 1 \text{amu} = 1.0003172 : 1
\]

The atomic mass unit previously used by the chemists, on the other hand, was one-sixteenth of the average atomic weight of natural oxygen consisting of the three isotopes \(^{16}\)O, \(^{17}\)O and \(^{18}\)O having the relative abundances 99.76%, 0.04% and 0.20% respectively. It may be noted that because of the mass-energy equivalence the disintegration energy with the change in mass due to the disintegration.

To obtain the value of the unit of atomic mass in \(^{12}\)C scale, we note that 1 mole of \(^{12}\)C has the mass of 12 g or \(12 \times 10^{-3}\) kg. Since 1 mole contains \(N_0\) atoms, where \(N_0 = 6.02205 \times 10^{23}\) is the Avogadro number, the mass of each \(^{12}\)C atom is

\[
12 \times 10^{-3}/N_0 \text{ or } 12 \times 1.660566 \times 10^{-27} \text{ kg}
\]

Hence the unit of atomic mass in \(^{12}\)C scale is

\[
1 \text{u} = \frac{12}{N_0} \times 1.660566 \times 10^{-27} \text{ kg}
\]

The energy-equivalent of this amount of mass is

\[
1 \text{u} = 1.660566 \times 10^{-27} \times c^2
\]

\[
= 1.660566 \times 10^{-27} \times 8.98755 \times 10^{16}
\]

\[
= 14.92447 \times 10^{-11} \text{ J}
\]

\[
= 14.92447 \times 10^{-11} \times 931.502 \text{ MeV}
\]

\[
= 0.005225 \times 931.502 = 4.87 \text{ MeV}
\]

The above calculation shows that the disintegration energy is less than one part in 40,000 of the mass of the disintegrating nucleus. So unless the masses of the atoms are determined with accuracies much better than the above it is not possible to correlate the measured disintegration energy with the change in mass due to the disintegration.

It may be noted that the study of nuclear disintegration energies provides direct experimental evidence in support of the mass-energy equivalence principle.

### Table 2.1

<table>
<thead>
<tr>
<th>Particle</th>
<th>Rest Mass in kg</th>
<th>Rest Mass (in u)</th>
<th>Rest Energy (eV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron (m_e)</td>
<td>9.10935 \times 10^{-31}</td>
<td>5.48580 \times 10^{-4}</td>
<td>5.11003 \times 10^5</td>
</tr>
<tr>
<td>Proton (M_p)</td>
<td>1.67265 \times 10^{-27}</td>
<td>1.0072765</td>
<td>9.38280 \times 10^8</td>
</tr>
<tr>
<td>Neutron (M_n)</td>
<td>1.67495 \times 10^{-27}</td>
<td>1.008650</td>
<td>9.39573 \times 10^8</td>
</tr>
</tbody>
</table>

The energy equivalents of the rest masses of the electron, proton and neutron, using the above conversion factor are given in Table 2.1.

#### 2.3 Importance of accurate determination of atomic masses

Atomic masses can be determined with accuracies better than one part in a million by modern mass spectrometers. Such great accuracies are needed for the determination of nuclear binding energies and in the calculation of nuclear disintegration energies.

As an example, consider the \(\alpha\)-disintegration of a heavy element like \(^{226}\)Ra \((Z = 88)\). As will be seen in Ch. IV, these nuclei spontaneously disintegrate by the emission of \(\alpha\)-particles of a few MeV kinetic energy. This energy is derived from the conversion of a part of the mass of the disintegrating parent nucleus into energy according to mass-energy equivalence relation.

The \(\alpha\)-disintegration of the nucleus of the \(^{226}\)Ra atom leads to the product nucleus \(^{222}\)Rn \((Z = 86)\) according to the equation

\[
^{226}\text{Ra} \rightarrow ^{222}\text{Rn} + ^4\text{He}
\]

The masses of the different atoms taking part in the above process are as follows:

\[
M\left(^{226}\text{Ra}\right) = 226.025436 \text{u}
\]

\[
M\left(^{222}\text{Rn}\right) = 222.017608 \text{u}
\]

\[
M\left(^4\text{He}\right) = 4.002603 \text{u}
\]

Then according to Eq. (4.5-2) the \(\alpha\)-disintegration energy is

\[
Q_\alpha = \left(M\left(^{226}\text{Ra}\right) - M\left(^{222}\text{Rn}\right) - M\left(^4\text{He}\right)\right) c^2
\]

\[
= (226.025436 - 222.017608 - 4.002603) \times 931.502
\]

\[
= 0.005225 \times 931.502 = 4.87 \text{ MeV}
\]

The above calculation shows that the disintegration energy is less than one part in 40,000 of the mass of the disintegrating nucleus. So unless the masses of the atoms are determined with accuracies much better than the above it is not possible to correlate the measured disintegration energy with the change in mass due to the disintegration.

It may be noted that the study of nuclear disintegration energies provides direct experimental evidence in support of the mass-energy equivalence principle.

#### 2.4 Systematics of nuclear binding energy

Accurate determination of the atomic masses shows that these are very close to whole numbers, which are actually the mass numbers of the atoms, when the masses are expressed in the units of atomic masses in the \(^{12}\)C scale. The same is also true if the atomic masses are expressed in \(^{16}\)O scale.

Considering the \(^{12}\)C scale, the atomic mass of \(^{12}\)C is exactly 12 u. The masses of all other atoms, though close to the corresponding mass numbers (integral), differ slightly from the latter.

The masses of a few atoms listed in Table 2.2 below will bear this out.
The table shows that for very light atoms with $A < 20$ and for very heavy atoms with $A > 180$, the atomic masses are slightly greater than the corresponding mass numbers. In between the above values of $A$, the atomic masses are slightly less than the corresponding mass numbers.

The departure of the measured atomic mass $M(A, Z)$ from the mass numbers $A$ is quite significant. The difference between $M$ and $A$ is known as the mass defect $\Delta M$:

$$\Delta M = M(A, Z) - A \quad \text{(2.4-1)}$$

For example, since the atomic mass of $^4\text{He}$ (4.002603 u) is slightly greater than the mass number 4, its mass defect is $0.002603$ u. On the other hand, $^{75}\text{As}$ has the atomic mass 74.921597 u, which is slightly less than the mass number 75. Its mass defect is $-0.078403$ u. Thus the mass defect can be both positive and negative. For very light and very heavy atoms, the mass defect is positive, while in the intermediate region it is negative (see Table 2.2).

The mass defect of an atom divided by its mass number is known as the packing fraction ($f$), a term introduced by F.W. Aston. Thus

$$f = \frac{\Delta M}{M} = \frac{M(A, Z) - A}{A} \quad \text{(2.4-2)}$$

In the last column of Table 2.2, the packing fractions of the different atoms are listed. $f$ has the same sign as $\Delta M$ and is positive for very light and very heavy atoms. It is negative for the atoms in the intermediate region.

From Eq. (2.4-2), we have

$$M(A, Z) = A (1 + f) \quad \text{(2.4-3)}$$

It is found that the packing fraction $f$ varies in a systematic manner with the mass number $A$. The nature of this variation is shown graphically in Fig. 2.1.

From the figure it is seen that for very light nuclei the packing fraction is positive and decreases rapidly with increasing $A$. It becomes negative for $A$ greater than about 20, attains a minimum (negative) at $A \sim 60$. It then rises slowly for higher $A$ and becomes positive again for $A$ greater than about 180.

This systematic variation of $f$ with $A$ can be understood from nuclear binding energy considerations.

If the binding energy $E_B$ of a nucleus $^{2}X$ defined by Eq. (2.2-3) is divided by the mass number $A$, we get the binding energy per nucleon in the nucleus, which is known as the binding fraction ($f_B$) and is given by

$$f_B = \frac{E_B}{A} = \frac{ZM_H + NM_n - M(A, Z)}{A} \quad \text{(2.4-4)}$$

Here we have assumed that the masses are expressed in energy unit so that $c^2$ on the r.h.s. of Eq. (2.4-4) has been omitted.

We can estimate the values of $f_B$ for a few typical cases, using the mass values given in Table 2.2.

For deuteron ($^2\text{H}$), since $Z = 1, N = 1$,

$$E_B(^2\text{H}) = M_H + M_n - M_{^2\text{H}} = (1.007825 + 1.008665 - 2.014102) \times 931.5 = 2.224 \text{ MeV}$$

$$f_B(^2\text{H}) = \frac{2.224}{2} = 1.112 \text{ MeV per nucleon}$$

For the $\alpha$-particle ($^4\text{He}$), since $Z = 2, N = 2$,

$$E_B(^4\text{He}) = (2 \times 1.007825 + 2 \times 1.008665 - 4.002603) \times 931.5 = 28.3 \text{ MeV}$$

$$f_B(^4\text{He}) = \frac{28.3}{4} = 7.075 \text{ MeV per nucleon}$$

Table 2.2

<table>
<thead>
<tr>
<th>Atom</th>
<th>Atomic Mass (u)</th>
<th>Mass Defect (u)</th>
<th>Packing Fraction (u)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^1\text{H}$</td>
<td>1.008665</td>
<td>+0.008665</td>
<td>-</td>
</tr>
<tr>
<td>$^1\text{H}$</td>
<td>1.007825</td>
<td>+0.007825</td>
<td>-</td>
</tr>
<tr>
<td>$^2\text{H}$</td>
<td>2.014102</td>
<td>+0.014102</td>
<td>+0.007051</td>
</tr>
<tr>
<td>$^4\text{He}$</td>
<td>4.002603</td>
<td>+0.002603</td>
<td>+0.0006507</td>
</tr>
<tr>
<td>$^{12}\text{C}$</td>
<td>12</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$^{16}\text{O}$</td>
<td>15.994915</td>
<td>-0.005086</td>
<td>-0.0003178</td>
</tr>
<tr>
<td>$^{31}\text{P}$</td>
<td>30.973764</td>
<td>-0.02636</td>
<td>-0.0008463</td>
</tr>
<tr>
<td>$^{39}\text{Co}$</td>
<td>58.933189</td>
<td>-0.066811</td>
<td>-0.0011324</td>
</tr>
<tr>
<td>$^{75}\text{As}$</td>
<td>74.921597</td>
<td>-0.078403</td>
<td>-0.0010454</td>
</tr>
<tr>
<td>$^{127}\text{I}$</td>
<td>126.90447</td>
<td>-0.09553</td>
<td>-0.0007522</td>
</tr>
<tr>
<td>$^{197}\text{Au}$</td>
<td>196.96554</td>
<td>-0.03346</td>
<td>-0.0001698</td>
</tr>
<tr>
<td>$^{226}\text{Ra}$</td>
<td>226.02543</td>
<td>+0.02543</td>
<td>+0.0001125</td>
</tr>
<tr>
<td>$^{238}\text{U}$</td>
<td>238.05082</td>
<td>+0.05082</td>
<td>+0.002135</td>
</tr>
</tbody>
</table>

Fig. 2.1. Packing fraction curve.
For $^{16}$O nucleus, since $N = 8$, $Z = 8$,
\[
E_b (^{16}\text{O}) = (8 \times 1.007825 + 8 \times 1.008665 - 15.994915) \times 931.5
\approx 127.62 \text{ MeV}
\]
\[f_b (^{16}\text{O}) = \frac{127.62}{16} = 7.98 \text{ MeV/nucleon}
\]

The binding fractions of the different nuclei represent the relative strengths of their binding. Thus $^1$H is very weakly bound, compared to $^4$He or $^{16}$O. The nature of variation of $f_b$ for the different nuclei with $A$ is shown graphically in Fig. 2.2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.2}
\caption{Binding fraction curve.}
\end{figure}

The following points about the variation of $f_b$ against $A$ are noteworthy:

(a) $f_b$ for the very light nuclei is very small and rises rapidly with $A$ attaining a value of $\approx 8$ MeV/nucleon for $A \approx 20$. It then rises slowly with $A$ and attains a maximum of $8.7$ MeV/nucleon at $A \approx 56$. For higher $A$, it decreases slowly. (b) For $20 < A < 180$, the variation of $f_b$ is very slight, so that it may be taken to be approximately constant in this region having a mean value of $\approx 8.5$ MeV/nucleon. (c) For very heavy nuclei ($A > 180$), $f_b$ decreases monotonically with the increase of $A$. For the heaviest nuclei, $f_b$ is about 7.5 MeV/nucleon. (d) For very light nuclei, there are rapid fluctuations in the values of $f_b$. In particular, peaks are observed in the $f_b$ vs. $A$ graph for the even-even nuclei $^4$He, $^{8}$Be, $^{12}$C, $^{16}$O etc., for which $A = 4n$ where $n$ is an integer. Similar, but less prominent peaks are observed at the values of $Z$ or $N$ equal to 20, 28, 50, 82 and 126. These are known as magic numbers (see Ch. IX).

The appearance of the peaks shows greater stability of the corresponding nuclei relative to the nuclei in their immediate neighbourhood.

The nature of the binding fraction curve is complementary to the nature of the packing fraction curve (Fig. 2.1). The reason for this can be understood as follows. If we write $M_H = 1 + f_H$ and $M_n = 1 + f_n$ where $f_H = 0.007825 u$ and $f_n = 0.008665 u$ are constant, then we have
\[
E_b = Z (1 + f_H) + (1 + f_n) - M (A, Z)
\]
\[= (Z + N) + f_H + f_n - A (1 + f)
\]
\[= A + f_H + f_n - A - \Delta M
\]
where $\Delta M = Af$. Hence we get
\[
E_b = \frac{Zf_H + Nf_n}{A} - \frac{\Delta M}{A}
\]
\[f_b = \frac{E_b}{\Delta M} = \frac{Zf_H + Nf_n}{A} - \frac{\Delta M}{A}
\]

The first term on the r.h.s. of Eq. (2.4-6) is almost a constant specially for lower $A$ when $Z = N = A/2$.

Thus $f_b$ increases or decreases as $f$ decreases or increases respectively. Hence the graphs of variation of $f$ and $f_b$ with $A$ have complementary appearances. Corresponding to the minimum in the graph of $f$ vs. $A$, there is a maximum in the graph of $f_b$ vs. $A$. Also the region of negative slope for low $A$ in the first case, corresponds to the region of positive slope in the second case. For higher $A$ on the other hand, the region of positive slope in the first case corresponds to the region of negative slope in the second.

With the help of the binding fraction curve it is possible to explain in a qualitative manner the reasons for the $\alpha$-disintegration of heavy nuclei as also of the energy release in nuclear fission and fusion processes. These will be discussed at appropriate places.

2.5 Nuclear size

We have seen (§ 1.4) that Rutherford's theory of $\alpha$-particle scattering gives us an idea about the smallness of the nuclear size. Later, Rutherford and his collaborators performed scattering experiments with relatively higher energy $\alpha$-particles and observed departure from Rutherford scattering formula at large angles; i.e., for small impact parameters $b$. When $b$ becomes comparable to the nuclear radius $R$, the $\alpha$-particle begins to feel the effect of the nuclear force. Since the Rutherford's scattering formula is deduced on the assumption that the force acting on the $\alpha$-particle is purely electrostatic, departures from Eq. (1.2-10) would be expected, when this is no longer true. Putting $b = R$ in Eq. (1.2-7), we then get the limiting angle of scattering $\theta_c$ above which the ratio of the measured scattering cross section $\sigma$ to that given by Rutherford's formula ($\sigma_0$) will be different from unity. Using Eq. (1.2-7), we get,
\[
\cot \frac{\theta_c}{2} = \frac{4\pi e_0 M v^2 R}{Ze^2}
\]
\[\therefore \frac{\sigma}{\sigma_0} = \frac{\theta_c}{Z \theta_c} \approx 1 - \theta_c \frac{Z}{Z \theta_c}
\]
By noting the limiting angle \( \theta \) above which anomalous scattering takes place \( (\sigma/\sigma_0 \neq 1) \), Rutherford estimated the values of the nuclear radius \( R \) for a few light elements e.g., magnesium. These were of the order of a few times \( 10^{-15} \) m.

These estimates of Rutherford were not very accurate. In later years more accurate methods for the measurement of the nuclear radius have been developed. It should be noted that when we talk of the nuclear radius, we assume that the nucleus has a spherical shape. This is expected because of the short range character of the nuclear force. However, small departures from the sphericity of certain nuclei have been observed. This is inferred from the existence of electric quadrupole moment of these nuclei which is zero for the spherical nuclei. The departure from sphericity is however small. We shall revert this topic in § 2.11.

In the above discussion, it has been assumed that the nuclear charge is uniformly distributed. Experiments show that this is very nearly so and the nuclear charge density \( \rho_c \) is approximately constant. Experimental evidences also show that the distribution of nuclear matter (i.e., protons and neutrons) is nearly uniform, so that the nuclear matter density \( \rho_m \) is also approximately constant. Since nuclear mass is almost linearly proportional to the mass number \( A \), this means that:

\[
\rho_m \sim A/V = \text{constant}
\]

i.e., the nuclear volume \( V \propto A \). Assuming a spherical shape of the nucleus with a radius \( R \), we then get

\[
V = \frac{4}{3} \pi R^3 \propto A
\]

or,

\[
R \propto A^{1/3}
\]

so that

\[
R = r_0 A^{1/3}
\]

where \( r_0 \) is a constant, known as the nuclear radius parameter.

It should be noted that the nuclear radius, as discussed above, is the radius of nuclear mass distribution. We may also talk about the radius of nuclear charge distribution. Since the nuclear charge parameter \( (\text{i.e., the atomic number}) Z \) is almost linearly proportional to the mass number \( A \) and the nuclear charge density \( \rho_c \) is approximately the same throughout the nuclear volume, the distribution of the nuclear charge \( +Ze \) should follow the pattern of nuclear mass distribution. Hence the charge radius and the mass radius of the nucleus may be expected to be very nearly the same. This is due to the strong attractive forces within the nucleus. There are strong evidences to show that this is very nearly the same for both types of nucleons, viz., the protons and the neutrons and hence their distributions within the nuclear volume follow the same pattern.

We now consider the potential energy diagram shown in Fig. 2.3 for a charged particle like a proton or an \( \alpha \)-particle, which is acted upon by the electrostatic repulsive force of the nuclear charge \( +Ze \) when it is outside the nucleus \( (r > R) \), while inside the nucleus \( (r < R) \) a negative potential due to the short range of the specifically nuclear force acts upon it. Here \( r \) is the distance from the nuclear centre. We assume, arbitrarily that electrostatic force is not effective inside the nucleus, while the nuclear force becomes zero at the nuclear surface \( r = R \).

Fig. 2.3 shows that the nucleus is surrounded by a Coulomb potential barrier \( V_c = \frac{Ze^2}{4 \pi \varepsilon_0 R} \) for an incident particle of charge \( Ze \) for \( r > R \). At the nuclear surface the barrier height is given by

\[
V_R = \frac{Ze^2}{4 \pi \varepsilon_0 R}
\]

For the uranium nucleus with \( Z = 92 \) and \( R = 8 \times 10^{-15} \) m, \( V_R = 16.5 \) MeV for a proton, while \( V_R = 33.1 \) MeV for an \( \alpha \)-particle taking \( r_0 = 1.3 \times 10^{-15} \) m.

Classically, a charged particle of energy \( E \) less that \( V_R \) cannot escape from the nucleus, nor can enter it from outside. However quantum mechanically, because of the uncertainty principle, the position of the particle within the nucleus is not so well-defined, so that there is a finite probability of the particle penetrating through the barrier if \( E < V_R \). If somehow the particle with an initial energy \( +E \) outside the nucleus reaches the point \( r = b \) where \( V_c = E \), then it will be repelled by the electrostatic force of the positive charge of the residual nucleus and will fly away from the latter. We have discussed about such tunnelling through the potential barrier in Ch. XI of Vol. I, which, as we have seen, can account for the \( \alpha \)-disintegration of the heavy nuclei (see also Ch. IV).

The radius \( R \), as defined above, is usually known as the potential radius, as distinct from the charge or mass radius discussed previously and is slightly larger than the latter.

The charge radius is the most directly measurable one. It can be determined by several methods of which the method based on the scattering of high energy electrons (> 100 MeV) is the most accurate (see...
The nuclear radius is usually expressed in units of $10^{-15}$ m which by international convention is known as the femtometer, abbreviated as fm though the unofficial name fermi is more often used.

The mean squared radius of nuclear charge distribution can be defined as follows:

$$<r^2> = \frac{\int_0^R r^4 \rho (r) \, dr}{\int_0^R 4 \pi r^2 \rho (r) \, dr} \quad ...(2.5-4)$$

where $\rho (r)$ is the nuclear charge density. For a uniformly charged sphere ($\rho =$ constant) of radius $R$, this gives (since $\rho = 0$ for $r > R$)

$$<r^2> = \frac{\int_0^R r^4 \, dr}{\int_0^R 4 \pi r^2 \, dr} = \frac{3}{5} R^2$$

so that $R^2 = \frac{5}{3} <r^2> \quad ...(2.5-5)$

### 2.6 Measurement of the charge radius

(i) Electron scattering experiment:

Scattering of high energy electrons by nuclei constitutes the most direct method of measuring the charge radius of the nucleus and the nature of the nuclear charge distribution. This is because there is no specifically nuclear force acting on the electrons. Only the Coulomb attractive force due to the nuclear charge acts on them. If the de Broglie wavelength of the electrons is small compared to the nuclear radius, then the electron scattering experiment can reveal many details of the nuclear charge distribution.

Now according to de Broglie's theory of wave-corpuscular duality, the wavelength of a relativistic electron of rest mass $m_0$, having the total energy $E > m_0 c^2$ is given by (see Ch. IX, Vol I)

$$\lambda = \frac{c \hbar}{e \sqrt{V (V + 2 m_0 c^2 / e)}}$$

where $e V = E$ is the kinetic energy of the electron, $e$ being its charge. Substituting the values of $c$, $\hbar$, $e$ and $m_0$ we get

$$\lambda = \frac{12.4 \times 10^{-15}}{\sqrt{V (V + 1.02 \times 10^6)}} \text{ Å}$$

where $V$ is in volts. For electrons of kinetic energy $E_k = 200$ MeV, $V = 200 \times 10^6$ volts, which gives

$$\lambda = 6.19 \times 10^{-15} \text{ Å}$$

and

$$\frac{\lambda}{2 \pi} = 10^{-15} \text{ m} = 1 \text{ fm}$$

This is considerably smaller than the radius of most nuclei.

This shows that the use of electrons of a few hundred MeV energy can reveal considerable details regarding nuclear charge distribution.

The pioneering experiments on the elastic scattering of electrons by nuclei were carried out by R. Hofstadter and his group at Stanford University in the U.S.A. using the linear accelerator (SLAC), providing electron beam with energy upto 550 MeV. Their experimental arrangement is shown in Fig. 2.4.

### Fig. 2.4.

High energy electron scattering experiment. A-Accelerator; B-Beam stopper; M₁, M₂-Deflecting magnets; S=Collimating slits; T-Scattering chamber; P-Spectrometer; C-Concrete shielding.

The high energy electron beam from the linear accelerator A is deflected by means of the magnet M₁, and collimated by the slit system S. The deflecting magnet M₂ then directs the beam on to the target inside the scattering chamber T. The elastically scattered beam of electrons is then analysed by the large magnetic spectrometer P.

The quantum mechanical expression for the differential scattering cross-section of a relativistic electron from a spin-less target at the centre of mass angle $\theta$ is given by

$$\sigma (\theta) = \sigma_M (\theta) \left( \frac{F (q)}{q} \right)^2 \quad ...(2.6-1)$$

where $\sigma (\theta)$ is the scattering cross section and $\sigma_M (\theta)$ is the Mott cross section of elastic scattering from a point charge $+Ze$ and is given by

$$\sigma_M (\theta) = \frac{12 \times 10^5}{8 \pi E^2} \cos^2 \frac{\theta}{2} \sin^4 \frac{\theta}{2} \quad (..2.6-2)$$
$E$ is the energy of the electrons in the C.M. system. Eq. (2.6-2) is valid for low $Z$ elements only. $F(q)$ is called the form factor which gives the ratio by which the scattering cross-section is reduced when the charge $+Ze$ is spread out over finite volume. Because of the destructive interference between the electron waves scattered from different parts of the target nucleus, $F(q) < 1$. Using the Born approximation method of quantum mechanics, it can be shown that

$$F(q) = \frac{1}{Ze} \int \rho(r) \exp\left(iqr\right) dr$$ \hspace{1cm} (2.6-3)

$$= \frac{4\pi}{Zeq} \int \rho(r) \sin qr dr$$

where

$$q = \frac{p-p'}{\hbar}$$ \hspace{1cm} (2.6-4)

is a measure of the momentum transfer $p - p'$ in elastic scattering.

$lq|l$ depends on the angle of scattering and is given by

$$lq|l = \frac{2p}{\hbar} \sin \frac{\theta}{2}$$ \hspace{1cm} (2.6-5)

$\rho(r)$ is the charge density within the nucleus and the exponential is a phase factor over the volume. The form factor $F(q)$ is obviously equal to the Fourier transform of the charge density. It can be determined directly by scattering experiment from the ratio $\sigma(\theta)/\sigma_{0}(\theta)$. Then by using the inverse Fourier transformation, it is possible to determine $\rho(r)$. This is possible if measurements are made at a sufficiently large number of angles $\theta$. When this is not possible, a form of the density distribution, has to be assumed and best fit with the experimental data obtained by suitably adjusting the parameters in the expression. A particularly suitable form for $\rho(r)$ is given by

$$\rho(r) = \frac{\rho_{0}}{1 + \exp\left[\left(r - R_{1/2}\right)/a\right]}$$ \hspace{1cm} (2.6-6)

This is known as the Fermi distribution. The parameters $R_{1/2}$ and $a$ are adjusted to get the best fit with the experimental data. The above density distribution has the form shown in Fig. 2.5.

![Fig. 2.5. Fermi distribution for the nuclear charge density.](image)

Obviously for $r = R_{1/2}$, $\rho = \rho_{0}/2$ where $\rho_{0}$ is the charge density at the centre ($r = 0$). Thus $R_{1/2}$ is the half-value radius. The parameter $a$ determines the skin-thickness of the nucleus, which is the thickness in which $\rho(r)$ falls from 0.9 $\rho_{0}$ to 0.1 $\rho_{0}$ at the nuclear surface. This comes out to be $a = 4.4 a$.

If we approximate the above distribution by a uniform charge distribution, then the corresponding equivalent radius can be written as

$$R = r_{0}A^{1/3}$$

where $r_{0} = 1.32 \times 10^{-15} \text{ m}$ for $A < 50$ and $r_{0} = 1.21 \times 10^{-15} \text{ m}$ for $A > 50$. This confirms that nuclear matter is distributed almost uniformly within the nuclear volume, if we assume that the mass and charge radii are equal.

The value of $a$ is taken to be the same for all nuclei:

$$a = 0.5 \times 10^{-15} \text{ m} = 0.5 \text{ fm}$$

The mass of experimental data so far collected shows that for the spherical nuclei with $A > 15$, the charge distribution has a core of uniform density, surrounded by a skin of thickness 2.3 fm. The radius of half the maximum density $R_{1/2} = 1.07A^{1/3} \text{ fm}$. For $4 \leq A \leq 15$, there is no uniform core and the density decreases steadily with increasing $r$. There is some indication that for all nuclei there is slight diminution in the density near the centre. Further, the charge density in the core region decreases somewhat as $Z$ increases.

As stated before, the distribution of nuclear matter is very similar to that of nuclear charge. In Figs. 2.6(a) and (b) we compare the nuclear charge and nuclear mass distributions for the three nuclei $^{16}_{8}$O, $^{109}_{47}$Ag and $^{208}_{82}$Pb. In Table 2.3 are shown the different parameters for nuclear matter distribution.

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>$R_{1/2}$ (fm)</th>
<th>$a$ (fm)</th>
<th>$R/A^{1/3}$ (fm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{16}_{8}$O</td>
<td>2.61</td>
<td>0.513</td>
<td>1.04</td>
</tr>
<tr>
<td>$^{109}_{47}$Ag</td>
<td>5.33</td>
<td>0.523</td>
<td>1.12</td>
</tr>
<tr>
<td>$^{208}_{82}$Pb</td>
<td>6.65</td>
<td>0.526</td>
<td>1.12</td>
</tr>
</tbody>
</table>

Assuming a uniform mass distribution, if we write $A = \frac{4\pi}{3} R^{3} \rho_{m}$ then the experimental data gives the radius of uniform mass distribution $R = 1.1A^{1/3} \text{ fm}$ and $\rho_{m} = 0.17 \text{ nucleon per fm}^{3}$. The nuclear mass density is approximately the same at the centre for all nuclei. It increases slightly with $A$ and tends to the limiting value of 0.17 nucleon / fm$^{3}$. 
(ii) **Muonic x-ray method:**

There are some alternative methods of determining the mean squared radius of the nuclear charge distribution. One of these is based on the study of the so called mesonic x-rays.

As we shall see in Ch. XVIII, there is a large number of unstable fundamental particles, both charged and neutral, which are observed in nature (usually in the cosmic rays) or can be produced in the laboratory in high energy interactions. One of these is the muon (previously called the $\mu$-meson). The muons carry one unit of electronic charge. Both positive and negative muons $\mu^+$ and $\mu^-$ are known. They are heavier than the electrons, having rest-mass about 207 $m_e$, where $m_e$ is the electronic mass. They are subjected to the same type of interaction with the nuclei as the electrons, so that only the electrostatic Coulomb force due to the nuclear charge acts on them. **Specifically nuclear force (strong interaction)** does not act on them.

![Graph of nuclear charge and nucleon distributions](image)

**Fig. 2.6.** Experimentally determined (a) nuclear charge and (b) nucleon distributions.

When a beam of $\mu^-$ is passed through matter, some of them are readily captured in electron-like orbits round the nuclei of the capturing atoms forming a muonic atom. The radii of the muonic orbits are however much smaller than the electronic orbits, being smaller by the factor $m_e/m_\mu \sim 1/207$.

We know from Bohr’s theory of the spectra of hydrogen-like atom that the radius of the $n$th electronic orbit is

$$ r_n = \frac{4 \pi \varepsilon_0 n^2 \hbar^2}{m_e Z e^2} $$

where $Ze$ is the nuclear charge. So the radius of the muonic orbit should be

$$ r_\mu = \frac{4 \pi \varepsilon_0 n^2 \hbar^2}{m_\mu Z e^2} $$

...(2.6-7)

Here it is assumed that the nuclear charge $e$ is concentrated at the centre. For a heavy element like gold ($Z = 79$), the radius of muonic K-orbit ($n = 1$) will then be

$$ r = \frac{m_e}{m_\mu} \times 0.529 \text{Å} = 3.23 \times 10^{-5} \text{Å} = 3.23 \times 10^{-15} \text{m} $$

This is much smaller than the radius of the gold nucleus which is

$$ R(\text{Au}) = r_0 A^{1/3} = 1.2 \times 10^{-15} \times (197)^{1/3} = 7 \times 10^{-15} \text{m} $$

Thus the muonic K - orbit may be expected to lie wholly inside the nucleus in the case of heavy atoms.

When the muon is captured by an atom, it passes from the loosely bound outer orbits into the more tightly bound inner orbits. During the process, electromagnetic radiation is emitted. However the energy of such radiation is much higher than in the case of electronic transitions. The energy of the $\mu^-$ in the $n$th orbit will be (see Ch. IV, Vol. I), on point nucleus assumption

$$ E_n = -\frac{m_\mu Z^2 e^4}{32 \pi^2 \varepsilon_0 n^2 \hbar^2} $$

...(2.6-8)

Thus in the K-orbit of the gold atom, the orbital energy of $\mu^-$ will be

$$ E(\text{Au}) = -13.6 \times \frac{m_\mu}{m_e} \cdot Z^2 = -17.6 \text{ MeV} $$

This shows that the radiation emitted in the transitions in a muonic atom will lie in the extremely short wavelength x-ray region. From a measurement of these x-ray energies it is possible to estimate the binding energies of the muon in different orbits. However the binding energy in a particular orbit will be greatly reduced if the nuclear charge is spread over a finite region, so that a part of the captured muonic wave function lies within the nucleus. As we have seen above, this is expected for the heavier nuclei. The above mentioned reduction in energy from that
expected for a point nuclear charge can be theoretically correlated to the mean squared radius of the nuclear charge distribution. As an example, for Pb atom, the transition $2p_{3/2} \rightarrow 1s_{1/2}$ results in the emission of c.m. radiation of energy 6.02 MeV while that expected on point nucleus hypothesis is 16.4 MeV. The calculations are usually made on the assumption of a specific nuclear charge distribution.

The nuclear radius parameter estimated from muonic x-ray measurements are in reasonable agreement with the electron scattering experiments:

$$r_0 = (1.15 \pm 0.03) \times 10^{-15} \text{ m} = 1.15 \pm 0.03 \text{ fm}$$

(iii) Mirror nucleus method:

The third method of estimating the charge radius of a nucleus is based on the study of the energetics in the $\beta^+$ transformation of the mirror nuclei.

Pairs of isobaric nuclei, such as $^1\text{H}$ and $^1\text{He}$, $^3\text{He}$ and $^3\text{Li}$ etc. are known as mirror nuclei. The proton number ($Z$) and the neutron number ($N$) in them are interchanged and differ by one unit, so that their mass number is $A = 2Z - 1$ where $Z$ is the atomic number of the first member of the pair, the other having the atomic number ($Z - 1$). The first member of the pair is usually $\beta^+$ active and undergoes $\beta^+$ transformation into the second.

As we shall see in Ch. IX all nuclear masses can be fairly well represented by a semi-empirical formula, known as the Bethe-Weizsäcker mass formula, which contains a term depending on the Coulomb repulsion between the protons. If the $\beta^+$ transformation energy ($Q^+_\beta$) is calculated using this formula, then $Q^+_\beta$ is found to vary linearly with $A^{2/3}$, the constant of proportionality depending on the value $r_0$, the nuclear radius parameter.

$r_0$ estimated from these studies is found to agree fairly well with those estimated by the other methods discussed earlier.

$$r_0 = (1.28 \pm 0.05) \times 10^{-15} \text{ m} = 1.28 \pm 0.05 \text{ fm}$$

The different methods of measurement of the charge radius give a mean value of the radius parameter $r_0 = (1.19 + 0.1A^{-1/3})$ fm. As can be seen, this is slightly dependent on $A$.

2.7 Measurement of potential radius

As stated in § 2.4, the specifically nuclear force is a strong short-range force. The potential from which this force is derived is thus of short range and has a steep slope at the edge of the nucleus. It owes its origin to the strong short range internucleon interaction (see Ch. XVII). There are evidences to indicate that this is independent of the nature (i.e., charge state) of the nucleons, so that the $p-p$ and $n-n$ forces are equal (charge symmetry). In addition, the $p-n$ force is also the same in the same quantum state ($'S'$): Obviously for a complex nucleus, the specifically nuclear interaction will extend up to a distance of the same order of magnitude as the range of the internucleon interaction beyond the radius $R_0$ of nuclear charge distribution. This is the radius shown in the potential energy diagram (Fig. 2.3) and is known as the potential radius, which is thus slightly larger than $R_0$. We discuss below two different methods of estimating the potential radius.

(i) Life time of alpha emitters:

Historically the earliest method of estimating the potential radius was based on the study of alpha-disintegration of heavy nuclei like $^{238}\text{U}$, $^{226}\text{Ra}$ etc. As discussed in Ch. XI Vol. I, alpha-disintegration of heavy nuclei takes place due to the penetration of the Coulomb potential barrier surrounding the nucleus. A more detailed theory is discussed in Ch. IV. According to this, the barrier penetration probability (transmission co-efficient) is given by

$$T = \exp (-G) \quad \ldots (2.7-1)$$

where

$$G = \frac{2}{\hbar} \left( \frac{M Ze^2 b}{\pi \epsilon_0} \right)^{1/2} \left( \cos \theta \sqrt{\frac{R}{b}} - \sqrt{\frac{R^2 - b^2}{b^2}} \right) \ldots (2.7-2)$$

where $R$ is the nuclear radius (potential radius) and $b$ is the distance from the centre to the point where the energy $E$ of the $\alpha$-particle is equal to the Coulomb potential energy $V_0 = 2Ze^2/4 \pi \epsilon_0 r$. Here $Z$ is the atomic number of the residual nucleus. $M$ and $2e$ are the mass and the charge of the $\alpha$-particle; $r$ is measured from the centre of the nucleus.

If $n$ be frequency of collision of the $\alpha$-particle against the nuclear wall inside the nucleus, then the probability of penetration through the barrier per second is $p = nT$. The reciprocal of this is the mean life of $\alpha$-decay which can be measured:

$$\tau_m = \frac{1}{p} = \frac{1}{nT} \quad \ldots (2.7-3)$$

Thus by measuring the mean life it is possible to estimate the potential radius $R$. Writing $R = r_0 \cdot A^{1/3}$ as before, the potential radius parameter is found to be $r_0 = 1.48 \times 10^{-15}$ m.

It should be noted that though the above theoretical formula does not reproduce the $\alpha$-decay life times accurately and may deviate by several orders of magnitude from the experimental value, it gives a much more precise estimate of the nuclear radius $R$, even from a rough knowledge of $\tau_m$.

$r_0$ estimated by this method is somewhat higher than that for the charge or mass radius parameter. A correction due to the finite radius of the $\alpha$ – particle ($R_0 = 1.2 \times 10^{-15}$ m) gives the radius of the residual nucleus $R_A$ such that $R = R_A + R_\alpha$ where $R_\alpha$ can be expressed by the formula

$$R_\alpha = r_0 A^{1/3}$$

The new parameter $r_{0A} = 1.4 \times 10^{-15}$ m \quad \ldots (2.7-4)
(ii) Neutron scattering experiments:

In these experiments, mono-energetic beams of fast neutrons are allowed to be scattered by nuclei. Since neutrons interact mainly by the strong specifically nuclear interaction with the nucleus, this method actually detects the edge of the nuclear potential well. It can be shown that the total cross-section for fast neutrons is given by (see Ch. XI)

$$\sigma_f = 2\pi (R + \lambda)^2 = 2\pi R^2$$

...(2.7-5)

where the de Broglie wavelength \(\lambda \ll R\), which happens at high energies, \(\lambda \approx \frac{\hbar}{m} v\). Also at such high energies, the neutron absorption cross section is given by

$$\sigma_a = \pi R^2$$

assuming a perfectly black nucleus which absorbs all the neutrons falling on it.

The measurements of the above cross-sections give a radius parameter \(r_0 = 1.25 \times 10^{-15}\) m.

Neutron measurements are usually difficult. So measurements using charged particles which interact strongly with the nuclei at close range, such as \(\alpha\)-particles or protons upto a few hundred MeV have also been made. In the \(\alpha\)-particle experiments, the critical angle of scattering at which deviations are observed from the Rutherford scattering is measured. \(\theta_c\) can be correlated with the critical distance at which the effect of the specifically nuclear force begins to be felt (see § 2.4).

In proton elastic scattering experiments (5 to 200 MeV), diffraction patterns are observed due to the extension of the potential beyond the nuclear edge. A specific form of the nuclear potential is assumed to fit the experimental data. The following potential form due to Woods and Saxon (optical potential) is usually employed to analyse the data (see Ch. XI):

$$V(r) = V_0 \frac{1}{1 + \exp \left(\frac{(r - R_1/\alpha)}{\alpha}\right)}$$

...(2.7-6)

This has a radial dependence similar to the Fermi charge distribution discussed in § 2.5. \(R_1/\alpha\) and \(\alpha\) have the same meanings as before.

A value \(r_0 = 1.33 \times 10^{-15}\) m is derived from the experimental data.

The potential radius is about 0.7 fm greater than the charge radius which may be taken to be the measure of the range of nuclear force.

We can summarize the results of the different types of measurements as below:

- (a) Mass distribution: \(r_m = 1.1 \times 10^{-15}\) m.
- (b) Equivalent square well for charge distribution:
  $$r_{eq} = (1.2 \text{ to } 1.3) \times 10^{-15}\text{ m}.$$
- (c) Optical potential: \(r_{ov} = 1.25 \times 10^{-15}\) m.

2.8 Nuclear spin

As stated before, a complex nucleus is made up of protons and neutrons, collectively known as nucleons. The protons and neutrons have intrinsic spin angular momentum \(\frac{1}{2}\) (in unit of \(\hbar\) ) each, just like the electrons. In addition, the nucleons also possess quantized orbital angular momenta about the centre of mass of the nucleus, like the electrons in the atom. The resultant angular momentum \(I\) of the nucleus is thus the vector sum of the orbital angular momentum \(L\) and spin angular momentum \(S\) of the nucleus:

$$I = L + S$$

...(2.8.1)

Quantum mechanical considerations show that the total orbital and spin angular momenta of the nucleus are given by

$$p_l^2 = I (I + 1) \hbar^2$$
$$p_s^2 = S (S + 1) \hbar^2$$
$$p_s^2 = S (S + 1) \hbar^2$$

During measurement, it is the largest component of the angular momentum along the direction of the applied electric or magnetic field which is determined. For the three cases mentioned above, these have the magnitudes \(I\), \(L\) and \(S\) respectively in units of \(\hbar\).

The resultant spin angular momentum of the nucleus is obtained by the vector addition of the spins of the individual nucleons: \(S = \sum s_i\). Similarly, the resultant orbital angular momentum is given by \(L = \sum l_i\).

Since \(s_i = \frac{1}{2}\), \(S\) can be either integral or half-integral, depending on whether the number of nucleons \(A\) in the nucleus is even or odd. On the other hand, since \(l_i\) is integral (0, 1, 2, etc.), \(L\) is integral or zero. Thus the total angular momentum \(I\) of the nucleus can be either integral (for \(A\) even) or half odd integral (for \(A\) odd). This is in agreement with observations.

The total nuclear angular momentum \(I\) is usually referred to as the nuclear spin. Measurements of the ground state spin of nuclei show that for even \(Z\) even \(N\) nuclei, the nuclear spin is invariably zero \((I = 0)\). This shows that there is a tendency of the nucleons inside the nucleus to form pairs with equal and oppositely aligned angular momenta, which cancel out in pairs for like nucleons.

Another important point to note is that the measured values of the ground state spins of the nucleus are small integers or half odd integers, the highest measured value being \(9/2\) which is small compared to the sum of the absolute values of \(l_i\) and \(s_i\) of all the individual nucleons contained in the nucleus. This is in conformity with what was stated above regarding pair formation within the nucleus. Majority of the nucleons of either type seems to form a core in which even numbers of protons and neutrons are grouped in pairs of zero spin and orbital angular momenta so that the core itself has zero total angular momentum. The few remaining nucleons outside the core determine the nuclear spin which is thus a small number, integral or half odd integral.

Methods of measurement of the ground state spins of nuclei will be discussed in § 8.10. Spins of excited states of nuclei are deduced from nuclear disintegration and nuclear reaction data.
Pauli's spin formalism

It may be mentioned here that the spin of a spin 1/2 particle like the nucleons moving non-relativistically is treated in terms of Pauli's theory. (See appendix AX Vol. I). Pauli introduced the spin operator \( \sigma \) related to the spin vector \( s \) through the relation \( s = \frac{\hbar}{2} \sigma \); \( \sigma \) has the three components \( \sigma_x, \sigma_y, \sigma_z \) which are 2 \( \times \) 2 matrices (Pauli matrices) as given below (see Introductory Quantum Mechanics, Second Edition, by the author).

\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

...(2.8-2)

Then \( \sigma_x^2 = \sigma_y^2 = \sigma_z^2 = 1 \) which is a 2 \( \times \) 2 unit matrix (1). The two states of the particle (spin up and spin down) are the two component Pauli spinors

\[
\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\]

...(2.8-3)

Operation of \( \alpha \) and \( \beta \) by the Pauli matrices gives the following results, as can be easily verified by direct matrix multiplication:

\[
\begin{align*}
\sigma_x \alpha &= \beta \\
\sigma_x \beta &= -\alpha \\
\sigma_y \alpha &= i \beta \\
\sigma_y \beta &= -i \alpha \\
\sigma_z \alpha &= \alpha \\
\sigma_z \beta &= -\beta
\end{align*}
\]

...(2.8-4)

We also have

\[
\sigma_x^2 \alpha = 3 \alpha, \quad \sigma_x^2 \beta = 3 \beta
\]

which gives

\[
\begin{align*}
s_x^2 \alpha &= \frac{3\hbar^2}{4} \alpha = \frac{1}{2} \left( \frac{1}{2} + 1 \right) \hbar^2 \alpha = s(s+1) \hbar^2 \alpha \\
s_x^2 \beta &= \frac{3\hbar^2}{4} \beta = \frac{1}{2} \left( \frac{1}{2} + 1 \right) \hbar^2 \beta = s(s+1) \hbar^2 \beta
\end{align*}
\]

...(2.8-5)

Also \( s_x \alpha = \frac{\hbar}{2} \alpha, \quad s_x \beta = -\frac{\hbar}{2} \beta \)

...(2.8-6)

Thus \( \alpha \) and \( \beta \) are the simultaneous eigen-vectors of \( s_x^2 \) and \( s_z \), belonging to the eigen-values \( \frac{3\hbar^2}{4} \) and \( \pm \frac{\hbar^2}{2} \) respectively, the plus sign corresponding to the spin up state and the minus sign corresponds to the spin down state.

The components of \( \sigma \) anti-commute, which means

\[
\sigma_x \sigma_y + \sigma_y \sigma_x = 0, \quad \sigma_x \sigma_z + \sigma_z \sigma_x = 0, \quad \sigma_y \sigma_z + \sigma_z \sigma_y = 0
\]

...(2.8-7)

We further have

\[
\sigma_x \sigma_y = \sigma_y \sigma_x = 2i \sigma_z, \quad \sigma_y \sigma_z = \sigma_z \sigma_y = 2i \sigma_x, \quad \sigma_z \sigma_x = \sigma_x \sigma_z = 2i \sigma_y
\]

...(2.8-8)

These give

\[
\sigma_x \sigma_y = i \sigma_z, \quad \sigma_y \sigma_z = i \sigma_x, \quad \sigma_z \sigma_x = i \sigma_y
\]

...(2.8-9)

2.9 Parity of nuclei

We discussed about the parity of the wave function in Vol. I. If we reflect the coordinate system at the origin, i.e., change from \( x, y, z \) to \(-x, -y, -z\), then the wave function of a physical system changes from \( \Psi(x, y, z) \) to \( \Psi(-x, -y, -z) \). It was shown in § 11.18 Vol. I, that if the Hamiltonian remains invariant under space inversion, then the changed wave function can be related to the original wave function in two different possible ways:

\[
\Psi(-x, -y, -z) = \pm \Psi(x, y, z)
\]

...(2.9-1)

or,

\[
\Psi(-x, -y, -z) = -\Psi(x, y, z)
\]

...(2.9-2)

In the first case the wave function is said to have even parity while in the second case it has odd parity.

Parity, as defined above, depends on the quantum mechanical state of motion of the system. As seen in Vol. I, in the case of a particle acted upon by a central force, the parity is determined by the azimuthal quantum number \( l \), being even for \( l = 0 \) or even and odd for \( l = \text{odd} \).

Apart from the orbital parity, fundamental particles may possess intrinsic parity. This is related to the inversion of some internal axis of the particle. It is actually defined in a relative manner. By convention, the nucleons are taken to have even parity. It is then fixed for the other particles in such a way that in an interaction between the particles involving strong nuclear or electromagnetic force, total parity must be conserved. By total parity we mean the product of the orbital and intrinsic parities (see Ch. XVIII for further discussion).

2.10 Statistics of nuclei

The concept of assigning statistics to fundamental particles is discussed in Ch. XX of Vol. I. In the case of a subatomic particle governed by the laws of quantum mechanics, the wave function of a system of two identical particles is either symmetric or anti-symmetric in the interchange of the coordinates of the two particles. If by such an interchange, the sign of the wave function does not change, we get a symmetric wave function and the resulting quantum statistics is known as Bose-Einstein statistics. In the reverse case, when the sign of the wave function changes as a result of the interchange, we have an antisymmetric wave function and the resulting statistics is called Fermi-Dirac statistics.

All fundamental particles (or their assemblies) can be grouped into two classes. The class of particles having spin 0 or integral obey B-E statistics and are known as bosons while particles having half-integral spins (\( \frac{1}{2}, \frac{3}{2}, \ldots \)) obey F-D statistics and are called fermions. In the latter case, the particles obey Pauli's exclusion principle, so that no two particles can occupy the same quantum state. In the first case (i.e. for bosons), the particles do not obey exclusion principle and any number of them can occupy the same quantum state.

Statistics is conserved in nuclear reactions.
2.11 Magnetic dipole moment of nuclei

Like the electron, the proton and the neutron possess intrinsic magnetic dipole moments. The measured values of the magnetic moments of the proton and the neutron are

\[ \mu_p = 2.7927 \mu_N \]
\[ \mu_n = -1.9131 \mu_N \]

where

\[ \mu_N = e\hbar / 2M_p \]

is called the nuclear magneton. \( e \) and \( M \) are the charge and mass of the proton. \( \mu_N \) is analogous to the Bohr magneton \( \mu_B = e\hbar / 2m_e \), which is the unit of atomic magnetic moment discussed in § 6.2 of Vol. I. \( \mu_N \) is much smaller than \( \mu_B \), being only 1/1836 part of the latter. Since

\[ \mu_B = 9.2849 \times 10^{-24} \text{ J/T} \]

The above numerical values show that the proton and neutron magnetic moments are of the order of \( 10^{-3} \) times the electronic magnetic moment, which is equal to the Bohr magneton (\( \mu_B = \mu_p \)). Since the nuclei are made up of protons and neutrons, the magnetic moments of the nuclei are also much smaller than the atomic magnetic moments, the latter being of the order of electronic magnetic moments.

Except for a minor correction, the electronic magnetic moment is correctly predicted by the relativistic quantum mechanical theory of the electron propounded by P.A.M. Dirac (see § 6.5, Vol. I). If the motion of the electron is described by the same theory, then the proton should have a magnetic moment \( \mu_p = \mu_n \). However this is not so, and \( \mu_p \) is greater than \( \mu_n \). Further the neutron being an uncharged particle, is not normally expected to have a magnetic moment. Again this is not true and \( \mu_n \) has a magnitude greater than \( \mu_n \). These anomalous values of the magnetic moments of the proton and the neutron can be understood, at least qualitatively, on the basis of the meson theory (see Ch. XVII).

It should be noted that the magnetic moments of the proton and the neutron are intimately related to their intrinsic spinangular momenta, which are given by

\[ p_p = s_p \hbar \]
\[ p_n = s_n \hbar \]

with \( s_p = s_n = \frac{1}{2} \). It was shown in § 6.3, Vol. I, that the ratio of the magnetic moment \( \mu_e \) to the spin angular momentum \( p_e \) of the electron is given by

\[ \frac{\mu_e}{p_e} = g_e \frac{e}{2m_e} \]

where \( p_e = s_e \hbar / 2 \), \( g_e \) being the Lande factor. It has the value \( g_e = -2 \).

The quantity of the r.h.s. of the above equation is the gyromagnetic ratio for the spin motion of the electron. The factor \( g_e = -2 \) was at first introduced by S. Goudsmit and G.E. Uhlenbeck on ad hoc basis but later found justification from the Dirac electron theory.

In the case of the proton and the neutron, we can write, in analogy with Eq. (2.11-2)

\[ \frac{\mu_p}{p_p} = \frac{e}{2M_p} \]

Substituting the values of \( p_p \) and \( p_n \), we get

\[ \frac{\mu_p}{p_p} = \frac{e}{2M_p} \cdot s_p \hbar = \frac{e}{2} \mu_N \]

Comparison with Eqs. (2.11-1) gives

\[ g_p = 2 \times 2.7927, g_n = -2 \times 1.9131 \]

Eqs. (2.11-5) can be written in vector forms (in nuclear magnetons) as

\[ \overrightarrow{\mu_p} = \frac{1}{2} \sigma_p \sigma_p \]
\[ \overrightarrow{\mu_n} = \frac{1}{2} \sigma_n \sigma_n \]

\( \sigma_p \) and \( \sigma_n \) are the Pauli spin operators.

The quantity \( g_e \) appearing in Eq. (2.11-2) is negative because of the negative sign of the electronic charge. Classically, the rotation of the electron constitutes a current opposite to the direction of its rotation. This current loop gives rise to a magnetic moment perpendicular to the plane of the loop directed opposite to the angular momentum associated with the rotation i.e. \( \mu_e \) is opposite to \( p_e \), so that \( g_e \) is negative.

For the proton, because of its positive charge, the directions of \( \mu_p \) and hence \( g_p \) is positive.

The negative sign of \( g_n \) then obviously indicates that \( \mu_n \) is directed opposite to \( p_n \).

For a complex nucleus, the intrinsic magnetic moments of all the protons are to be vectorially added to give the resultant \( \Sigma \mu_{p} \). Similarly the intrinsic magnetic moments of all the neutrons are to be vectorially added to give the vector \( \Sigma \mu_{n} \). In addition, the orbital rotations of the protons will also contribute to the net magnetic moment of the nucleus equal to \( \Sigma (L^2_{p}) \). This last can be defined in the same way as in the case of the orbital motion of the electron. If \( L_{p} \) be the resultant orbital angular momentum due to the orbital motion of the protons, then we can write

\[ \frac{\mu_L}{p_L} = \frac{e}{2M_p} \]
for a nucleus of atomic number $Z$ will be due to the electric dipole moment in the nucleus. Its magnitude is given by

$$p = qd.$$  

Actually the protons and neutrons, which are electrically neutral, give a contribution to the electric dipole moment of the nucleus of different orders located at the origin. Thus the first term corresponds to the potential due to an electric monopole $+e$, $Z$ being the atomic number. The second term in the expansion is an odd function, since $z'$ is an odd function. Hence the integral vanishes.

As discussed in § 2.8, the protons and the neutrons tend to form pairs with oppositely aligned spins, giving a resultant spin $0$. Obviously such pairs will also have zero magnetic moments. Hence the net magnetic moment of the nucleus will be determined by the nucleons outside the even $Z$, even $N$ core for which the net magnetic moment is zero. As in the case of the nuclear spin, this makes the value of the magnetic moment of the nucleus comparable to the proton or neutron magnetic moments.

The experimental methods for the determination of the magnetic dipole moments of nuclei will be discussed in Ch. VIII.

2.12 Electric moments of nuclei

The atomic nucleus is a positively charged body of finite dimensions. As is well known, the electrostatic potential $\Phi(r, \theta)$ due to an azimuthally symmetric distribution of electric charges can be expanded in ascending powers of $1/r$, where $r$ is the distance of the point where the potential is measured from the origin of the coordinate system.

$$\Phi(r, \theta) = \frac{1}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r_n} P_n(\cos \theta).$$  

where $P_n$'s are the Legendre polynomials of different orders. The different terms in the expansion correspond to potentials due to electric multipoles of different orders located at the origin. Thus the first term corresponds to the electric dipole moment of the nucleus which is nothing but a point charge $+Ze$, $Z$ being the atomic number. The second term in the expansion corresponds to the potential due to an electric dipole.

It is known that the electric dipole moment of a system of two equal and opposite charges $\pm q$ separated by an infinitesimal distance $d$ is given by $p = qd$. Since the nucleus is made up of positively charged protons (each of charge $+e$) and neutrons which are electrically neutral, a displacement of the centres of mass of the two types of particles will cause an electric dipole moment to appear in the nucleus. Its magnitude for a nucleus of atomic number $Z$ will be $p = Zed$. Actually the protons and neutrons are distributed throughout the nuclear volume. Assuming a continuous distribution of the nuclear charge with a density $\rho(r')$, the projection of the dipole moment along the $z$-axis is given by

$$p_z = \int \rho(r') d\tau'$$  

where the integration is evaluated over the whole volume of the nucleus.

The charge density $\rho (r')$ can be written as

$$\rho (r') = \sum_{i=1}^{Z} P_i (r')$$  

where $P_i (r')$ is the probability of the $i$th proton being at $r'$; $P_i$ can be written in terms of the wave function $\Psi (r_1', r_2', ... r_i')$,

$$P_i (r') = \int |\Psi (r_1', r_2', ... r_i')|^2 d\tau'_i$$  

where $d\tau'_i = dr'_1 dr'_2 ... dr'_i$ excluding $d\tau'_i$. The integration is carried out over the coordinates of all the nucleons except the $i$th.

Hence we have

$$p_z = e \sum_{i=1}^{Z} \int z'_i | (r_1', r_2', ... r_i')|^2 d\tau'_i$$  

where $d\tau'$ is the volume element for the coordinates of all the nucleons including the $i$th, i.e.

$$d\tau = d\tau'_1 d\tau'_2 ... d\tau'_i ... d\tau'_N.$$  

Now according to the law of conservation of parity, $|\Psi (r_1', r_2', ... r_i')|^2 = |\Psi (-r_1', -r_2', ... -r_i')|^2$.

So the integrand in Eq. (2.12-5) is an odd function, since $z'_i$ is an odd function. Hence the integral vanishes.

Thus we get the important result that the electric dipole moment of a nucleus in its ground state vanishes. This is also the case for all non-degenerate excited states of the nucleus. The above argument also holds for static electric moments of all odd orders (e.g., octupole moment) which are therefore all zero for the nucleus.

Similarly it can be shown that the static magnetic moments of even orders are all zero for the nucleus.

Electric quadrupole moment

The third term in the expansion (2.12-1) corresponds to the potential due to the electric quadrupole moment $Q$ of the given cylindrically symmetric charge distribution located at the origin of the co-ordinate system.

The simplest system of charges which has electric quadrupole moment can be generated by the displacement of an electric dipole with respect to itself with the sign of the dipole moment $p$ reversed, as shown in Fig. 2.7a. Its quadrupole moment is given by $Q = 2qdd'$. Since $d$ and $d'$ are both measured in units
of length, \(dd'\) has the dimensions of an area (metre\(^2\)) which is also taken to be the unit of \(Q\).

Fig. 2.7. Electric quadrupole moments for different charge distributions. (a) Two dipoles oppositely aligned. (b) Spherical charge distribution \((Q_0 = 0)\). (c) Prolate spheroid \((Q_0 > 0)\). (d) Oblate spheroid \((Q_0 < 0)\).

We distinguish between the intrinsic quadrupole moment of a nucleus \(Q_0\) from the observed quadrupole moment \(Q\). \(Q_0\) is defined by the relation:

\[
Q_0 = \frac{1}{e} \int (3\text{z'}^2 - r'^2) \rho (r') \, dr' \tag{2.12-6}
\]

where the integration is carried out over the whole volume of the nucleus. \(r' (x', y', z')\) is measured from the centre of mass of the nucleus. The nucleus is assumed to have a symmetry axis along \(z'\); \(e\) is the charge on each proton. The expression for \(Q_0\) is divided by \(e\) so that its unit is (length\(^2\)).

\(Q_0\) is usually measured in barns:

1 barn = 10\(^{-28}\) m\(^2\).

For a spherically symmetric charge distribution

\[
\int \rho (r') x'^2 d\tau' = \int \rho (r') y'^2 d\tau' = \int \rho (r') z'^2 d\tau' = \frac{1}{3} \int \rho (r') r'^2 d\tau' \tag{2.12-7}
\]

Eq. (2.12-6) then makes \(Q_0 = 0\) for a spherical nucleus for which \(l = 0\) (see Fig. 2.7 b).

Thus non-zero quadrupole moment gives a measure of the departure of the nuclear shape from sphericity.

For non-spherical nuclei, we distinguish between the intrinsic quadrupole moment \(Q_0\) and observed quadrupole moment \(Q\). \(Q_0\) is measured in a reference frame, which is fixed to the nucleus. \(Q\) on the other hand, is measured in a reference frame fixed in the laboratory. The two are related by Eq. (2.12-13) given later.

Eqs. (2.12-6) and (2.12-7) show that for a nucleus in the shape of a prolate spheroid, elongated along the \(z'\) axis (cigar-shaped), \(Q_0 > 0\) (see Fig. 2.7 c) since in this case

\[
\int \rho (r') z'^2 d\tau' > \frac{1}{3} \int \rho (r') r'^2 d\tau' \tag{2.12-8}
\]

On the other hand for a nucleus in the shape of an oblate spheroid (pancake-shaped)

\[
\int \rho (r') z'^2 d\tau' < \frac{1}{3} \int \rho (r') r'^2 d\tau'. \tag{2.12-9}
\]

So \(Q_0 < 0\) for such nuclei (see Fig. 2.7 d).

Measurement gives for the deuteron, \(Q_0 = + 2.82\) milli-barns (mb) = + 2.82 \(\times\) 10\(^{-31}\) m\(^2\). This shows that the charge distribution in the \(^2\)H nucleus has the shape of a prolate spheroid.

Quadrupole moment was first discovered in the deuteron from observations on the hyperfine structure of the atomic spectral lines. The interaction of the nuclear electric quadrupole moment with the homogeneous electric field due to the atomic electron distribution produces additional hyperfine splitting, which departs from the interval rule followed by the normal hyperfine splitting due to the interaction between the nuclear magnetic moment and the atomic magnetic moment (see Ch. VIII).

Quantum mechanically \(Q\) is defined as the expectation value \(Q_0\) of \(Q\) determined by the expectation value of \((3z'^2 - r'^2)\) for a given charge distribution for the state \(M_f = l\) where \(M_f\) is the projection of \(l\) along a given direction \(z\) in space. Obviously \(M_f = l\) is the maximum projection of \(l\) along the given direction. Since the magnitude of \(l^2\) is \(l(l + 1)\), the vector \(l\) can never align along \(z\).

As we shall see, the above value of \(Q_l\) vanishes for \(l = 0\) and \(l = 1\). The measured values of \(Q_l\) are found to vary from \(-8\) barns for \(^{123}\) Lu (\(Z = 71\)) to \(-1.0\) barn for \(^{123}\) Sb (\(Z = 51\)).

Quadrupole moment for an ellipsoidal charge distribution:

Let \(z'\) be the axis of symmetry of the ellipsoid (see Fig. 2.7 a) which has the semi-axes \(a\) and \(b\) as shown in the figure. Assuming uniform charge distribution with density \(\rho\), we have for a nucleus of charge \(Ze\).

\[
\rho = Ze / \frac{4}{3} \pi ab^2
\]

\[
\int \rho (r') z'^2 d\tau' < \frac{1}{3} \int \rho (r') r'^2 d\tau'. \tag{2.12-9}
\]

So \(Q_0 < 0\) for such nuclei (see Fig. 2.7 d).

Measurement gives for the deuteron, \(Q_0 = + 2.82\) milli-barns (mb) = + 2.82 \(\times\) 10\(^{-31}\) m\(^2\). This shows that the charge distribution in the \(^2\)H nucleus has the shape of a prolate spheroid.

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As we shall see, the above value of \(Q_l\) vanishes for \(l = 0\) and \(l = 1\). The measured values of \(Q_l\) are found to vary from \(-8\) barns for \(^{123}\) Lu (\(Z = 71\)) to \(-1.0\) barn for \(^{123}\) Sb (\(Z = 51\)).

Quadrupole moment for an ellipsoidal charge distribution:

Let \(z'\) be the axis of symmetry of the ellipsoid (see Fig. 2.8 a) which has the semi-axes \(a\) and \(b\) as shown in the figure. Assuming uniform charge distribution with density \(\rho\), we have for a nucleus of charge \(Ze\).

\[
\rho = Ze / \frac{4}{3} \pi ab^2
\]
The equation for the ellipsoid is \( \frac{x^2}{b^2} + \frac{y^2}{a^2} + \frac{z^2}{c^2} = 1 \).

In cylindrical polar coordinates \((r, \phi, z)\) shown in Fig. 2.8 \(b\) the equation reduces to

\[
\frac{s'^2}{b^2} + \frac{z'^2}{a^2} = 1
\]

where \(s'^2 = x'^2 + y'^2\). We thus have

\[
s'^2 = b^2 \left(1 - \frac{z'^2}{a^2}\right)
\]

The quadrupole moment in the body fixed coordinate system is

\[
Q_0 = Q_{zz} = \frac{1}{\varepsilon} \iiint p (3z'^2 - r'^2) \, d\tau
\]

\[
= \frac{3Z}{4\pi ab^2} \iiint (3z'^2 - x'^2 - y'^2 - z'^2) \, d\tau
\]

\[
= \frac{3Z}{4\pi ab^2} \iiint (2z'^2 - s'^2) \, d\sigma s' \, ds'
\]

\[
= \frac{3Z}{4\pi ab^2} \int d\sigma \int_{0}^{b} \int_{0}^{2\pi} (2z'^2 - s'^2) s' \, ds' \, d\phi
\]

\[
= \frac{2Z}{5} \left( a^2 - b^2 \right)
\]

Writing the eccentricity parameter \(\varepsilon = (a^2 - b^2)/(a^2 + b^2)\) and the mean squared radius of the nucleus \(<R^2> = (a^2 + b^2)\), we then get

\[
Q_0 = \frac{4}{3} Ze <R^2>
\]

Thus substituting the value of \(<R^2>\) we can find the value of the eccentricity parameter \(\varepsilon\) which is usually quite small, being -0.01 to 0.02. For nuclei with \(A = 150\) to 190 and \(A > 220\), considerable deformation is observed, \(\varepsilon\) being 0.1 to 0.2 (see Ch. IX).

**General case:**

We now consider the general case of a deformed nucleus with an axis of symmetry \(z\). If \(z\) denotes the space fixed axis, then the orientation of the nucleus in space is determined by the orientation of the nuclear spin \(I\) w.r.t. \(z\) and by its projection \(K\) upon the nuclear symmetry axis. From Fig. 2.9, it is seen that \(I = K + R\) where \(K\) is the projection of the vector sum \(\Sigma j_i\) of the total angular momenta of all the nucleons on the symmetry axis. \(R\) is the rotational angular momentum of the nucleus as a whole.

Quantum mechanical calculations give the following expression for the observed quadrupole moment \(Q\) which corresponds to \(M_j = I\) (see appendix AI):

\[
Q = \frac{3K^2 - I(I + 1)}{(I + 1)(2I + 3)} Q_0
\]

Eq. (2.12-12) shows that (a) \(Q\) is always less than \(Q_0\); (b) For \(I(I + 1) > 3K^2 \) \((\text{large } I)\), the sign of \(Q\) is opposite that of \(Q_0\); (c) For the nuclear ground state for which the nuclear rotational angular momentum \(R = 0\), so that \(I = K\), we have

\[
Q = -\frac{I(2I - 1)}{(I + 1)(2I + 3)} Q_0
\]

Thus for \(I = 0\) \((\text{even-even nuclei})\) or for \(I = \frac{1}{2}, \frac{3}{2}\) \((Q = 0\) even if \(Q_0 \neq 0\). For \(I > 1\), \(Q\) is non-zero if \(Q_0 \neq 0\). For \(I = 1\), \(|Q/Q_0| = 0.1;\) for \(I \gg 1\), \(Q/Q_0 \sim 1\).

The intrinsic quadrupole moment \(Q_0\) can be determined from the cross-section of Coulomb excitation of the rotational levels of deformed nuclei and the probability of \(\gamma\)-transition between such levels (see § 6.4).

### 2.13 Isospin

The neutron and the proton are regarded as two different states of the same entity, called the *nucleon*, differing only in their electrical charge. There are strong experimental evidences to show that the basic force (strong interaction) between two neutrons, as also between two protons, within the nuclei, is the same (see § 17.17). This of course does not take into account the electrostatic repulsion between the protons. Symbolically we can express this fact by writing (see Ch. XVII)

\[
(n - n) = (p - p)_{\text{soc}}
\]

\[
(n - n) = (p - p)_{\text{soc}} = (p - n)
\]

This is known as the charge symmetry of the nuclear force. In addition, the force between the neutron and the proton in \(s\) state is also equal to the above two forces, a fact known as the charge independence of the nuclear force. Symbolically we can therefore write

\[
(n - n) = (p - p)_{\text{soc}} = (p - n)
\]

Thus there is a basic symmetry of the nuclear forces which is broken only by the electromagnetic interaction due to the charge on the proton.

The situation is analogous to the case of a spin \(1/2\) particle which can exist in two different spin states determined by the parallel or antiparallel orientation of the spin vector \(s\) w.r.t. some specified direction in space. In the absence of any other interaction, these two spin states having the spin components \(s_z = +\frac{1}{2}\) and \(-\frac{1}{2}\) must have the same energy, i.e., the
Hamiltonian is invariant under the rotation of the spin vector $s$. This symmetry is broken by the presence of the magnetic field which splits up the two states which have different energies in the magnetic field.

Because of the formal analogy between the two possible charge states of a nucleon and the two possible spin states of a spin 1/2-particle, we can similarly introduce the concept of iso-spin (also known variously as isotopic spin, isobaric spin or simply $i$-spin) denoted by $i$ having the value 1/2. Then in an abstract isospin space the vector $i$ can have the two components $i_3 = \pm 1/2$. We take the $f$-component in the isospin space exactly the same as the Pauli spin operators $\sigma_x, \sigma_y$ and $\sigma_z$ respectively. The $x, y, z$-components of ordinary spin for the individual nucleons are added algebraically to give the resultant third component $T_3$ of the nucleus: $T_3 = \sum_i t_3 = \sum_i (p_3 + n_3) = \frac{Z + N}{2}$.

As the component cannot exceed the magnitude of the vector, we must have for the magnitude of the vector $T$

$$T \leq \sqrt{\frac{N - Z}{2}}$$

Since for a given $A$ (given isobaric multiplet), there may be different combinations of $N$ and $Z$ values, $T_3$ will be different for the different nuclei for a given value of $A$. Examples are the isospin multiplet $^{16}_6\text{C} = ^{16}_7\text{N}_3$ and $^{16}_8\text{O}_6$ at $A = 14$. Though there may be other possible combinations of $N$ and $Z$ for this particular $A$, these are the only three nuclei known to exist in nature or are artificially produced. So in this case, the values of $T_3$ are $-1, 0$ and $+1$ respectively for $^{16}_6\text{C}, ^{16}_7\text{N}_3$ and $^{16}_8\text{O}$. Hence we can write $T = 1$. The isospin multiplicity is given by $2T + 1$ which in the present case is 3. So we have the above triplet.
3.1 Discovery of radioactivity

The nuclei, for which the ground state properties were discussed in Chapter II, are known as stable nuclei. Once they are formed out of the requisite numbers of neutrons and protons in the particular quantum state (ground state), they remain in that state for an indefinite period of time, unless otherwise disturbed.

However, there is another class of nuclei, and they are more numerous, which are unstable. They break-up (disintegrate) spontaneously with the emission of some corpuscular or electromagnetic radiation of very high energy. In the first case, either the atomic number $Z$ or the mass number $A$ or both, of the nucleus change, thereby producing an altogether new nucleus. In the second case, the nucleus makes a transition from a quantum state of higher energy to one of lower energy. This phenomenon of spontaneous transformation of a nucleus is known as radioactivity which was the first nuclear phenomenon to be discovered. The credit for this discovery goes to the French physicist Henri Becquerel (1896).

Radioactivity was discovered almost by accident, just like the $x$-rays by Röntgen a year before. It was observed soon after the discovery of $x$-rays, that these rays produced fluorescence in various substances including glass. Becquerel, who had inherited from his father A. C. Becquerel an interest in the phenomena of phosphorescence and fluorescence (see Ch. IV in Vol. I), on hearing about the discovery of $x$-rays, wanted to find out whether the two phenomena were related. His first experiments gave negative results: the phosphorescent or fluorescent substances he tested did not emit $x$-rays.

Later, Becquerel repeated his experiments with a uranium salt (uranyl potassium sulphate). When a layer of this phosphorescent substance was placed on the thick black paper enwrapping a photographic plate and the whole thing was exposed to the sun light for several hours, it was found that on developing the plate, a silhouette of the phosphorescent substance appeared in black on the negative. From this he concluded that the phosphorescent substance in question emitted...
radiations which penetrated the back paper wrapping the photographic plate which was opaque to ordinary light.

The next few days after this discovery, were cloudy and failing to expose the phosphorescent substance to the sun light, he put everything in a dark drawer, leaving the sample of the uranium salt in place over the wrapped plates. When these plates were then developed (March 1, 1896), the silhouette appeared as before with great intensity. This showed at once that the uranium salt emitted rays capable of penetrating black paper, whether or not it had been previously exposed to sun light. Obviously the emission of the rays had nothing to do with the phenomenon of phosphorescence or fluorescence.

The uranium salt emitted the rays spontaneously, a phenomenon known as radioactivity.

Within a few days after this great discovery, Becquerel found that the radiation emitted by the uranium salt could ionize gases making them conducting, in addition to their ability to blacken photographic plates.

Thus it was possible to measure the "activity" of the sample (i.e., the intensity of the rays emitted) by measuring the ionization it produced using a simple gold-leaf electroscope.

Becquerel found evidence for the emission of the rays by different uranium salts. He also observed that the rays were emitted by powdered uranium metal. From these experiments he concluded that the penetrating radiation was emitted by the element uranium itself.

Soon after Becquerel's pioneering work, Madam Curie and her husband Pierre Curie started work on the newly discovered field of radioactivity. Apart from uranium, they investigated radioactivity of other elements. One very important fact which they observed was that the nature and the intensity of the radiations emitted by radioactive substances were not influenced by any physical or chemical change. Application of heat or pressure had no effect on radioactivity. As we have seen, chemical combination with other elements also had no effect on radioactivity. This fact shows that radioactivity of elements like uranium has nothing to do with the electrons revolving in the atomic orbits. For, as we know at present, chemical changes occur due to the rearrangement of the orbital electrons in the atoms. Later researches have established that the origin of radioactivity is associated with the atomic nucleus.

Madam Curie and Pierre Curie discovered the new radioactive elements polonium (Z = 84) and radium (Z = 88) which were more active than uranium. We shall talk about these discoveries later. With these discoveries, powerful sources of radioactive substances could be prepared, which completely revolutionized the new science of radioactivity.

Apart from the above named scientists in France, another very active group of scientists started investigations in this new field under the guidance of Ernest Rutherford (later Lord Rutherford) first at Cambridge, then at McGill University in Montreal, Canada and finally again at Cambridge.

Rutherford showed quite early that uranium emitted two kinds of rays. One of these was easily absorbed which Rutherford named α-rays. The other was more penetrating and was called β-rays. Later a third type of rays, called γ-rays, was discovered by the French scientist P.V. Villard (1900), which was found to be much more penetrating than the α or β rays.

As stated above, radioactivity was also discovered in elements other than uranium (Z = 92), e.g., thorium (Z = 90) and radium (Z = 88). These are called radioactive elements. It may be mentioned here that radioactivity is observed not only amongst a few naturally occurring substances as above. Large numbers of radioactive isotopes of elements which are not normally radioactive, have been produced in the laboratory in later years. We shall revert to the discussion of these artificially radioactive substances in Ch. X.

3.2 Radioactive disintegration and displacement law

It was mentioned in the previous section that radioactivity is a nuclear phenomenon. The nuclei of atoms spontaneously disintegrate to produce other nuclei by the emission of the corpuscular radiations, e.g., α or β rays. Rutherford proved that the α-rays were nothing but the nuclei of the helium atoms \(^{4}\text{He}\) having the mass number \(A = 4\) and the atomic number \(Z = 2\). Hence the emission of an α-particle from a nucleus \(^{A}\text{X}\) (parent nucleus) reduces the mass number by 4 units and the atomic number by 2 units, so that a new nucleus \(^{A-4}\text{Y}\) (daughter nucleus) is formed as a result of the α-disintegration process:

\[
^{2}\text{X} \xrightarrow{\alpha} ^{2-4}\text{Y}
\]  

Various experiments have established that the β-particles are very high energy electrons. Since they carry one unit of negative charge, the positive charge of the nucleus is increased by one unit as a result of β-particle emission. Thus the atomic number \(Z\) of the parent nucleus is increased to \(Z + 1\) in the daughter nucleus in this case. The modern view of the β-disintegration process is that the neutron inside the nucleus is spontaneously transformed into a proton. So the total number of neutrons and protons in the nucleus which is equal to its mass number \(A\) remains unchanged. Thus the nucleus \(^{A}\text{X}\) is changed into the nucleus \(^{A+1}\text{Y}\) due to β-disintegration:

\[
^{2}\text{X} \xrightarrow{\beta} ^{2+1}\text{Y}
\]

γ-rays are very high energy electromagnetic radiation, so that their wavelengths are very short. In many cases their wavelengths are even
shorter than those of x-rays. α or β disintegration of a nucleus is usually followed by the emission of one or more γ-ray photons.

γ-rays are also emitted during various types of nuclear transmutations induced by artificial means (see Ch. X). Evidently, the emission of γ-rays does not produce any fundamental change in the nature of the nucleus. Only the nucleus makes a transition from one energy state to another.

As early as 1913, two scientists F. Soddy and Fajans, observed the above changes in the values of $A$ and $Z$ during α—disintegration as also the change in $Z$ during β—disintegration. They systematized their observations in the form of an empirical law, known as the displacement law. The law may be stated thus: Alpha disintegration results in the reduction of the mass number of the atom by 4 units and its displacement in the periodic table by two steps to the left; on the other hand, beta disintegration leaves the mass number of the atom unchanged while its position in the periodic table is shifted by one step to the right.

When the law was formulated, our present day knowledge regarding the neutron-proton structure of the atomic nuclei was lacking. In course of time when these ideas became crystallized, the displacement law could be clearly explained, as we have seen above.

### 3.3 Growth and decay of radioactivity

The radioactivity of the elements uranium and thorium do not seem to change with time. On the other hand, there are some radioactive substances (e.g., radon gas) whose radioactivity decreases with time during the periods of observation i.e., the intensity of the radiation emitted by these substances decreases with time in a regular manner. Ultimately their capacity for the emission of radioactive radiations disappears altogether.

The British scientist William Crookes observed (1900) that there was some unknown substance present in the uranium salt which, when chemically precipitated with iron-hydroxide from the latter, carried away the entire radioactivity of the uranium salt. The uranium salt left after the chemical separation lost its radioactivity altogether. Crookes named the unknown substance thus precipitated as UX (uranium-X) and showed that its chemical properties were different from those of uranium. He then left the inactive uranium salt and the active precipitate as such for a few weeks and found that the uranium salt had regained its radioactivity as before, while the precipitate had become completely inactive.

Crookes showed that the above growth and decay of radioactivity of U and UX respectively could be represented by two mathematical formulas. If we write the activity (i.e., the intensity of the emitted radioactive radiations) as $A$, then the decay of the radioactivity of UX could be expressed as:

$$ A_X = A_0 \exp (-\lambda t) \quad \text{(3.3-1)} $$

On the other hand, the growth of the activity of U could be represented by the formula:

$$ A_U = A_0 \left[ 1 - \exp (-\lambda t) \right] \quad \text{(3.3-2)} $$

$A_0$ and $\lambda$ are time constants; $t$ denotes the time after the separation of UX from U. Thus the decay and growth of radioactivity follow exponential laws. $A_0$ is the activity of UX immediately after its separation from U. Eqs. (3.3-1) and (3.3-2) show that after a long time ($t \to \infty$) compared to $1/\lambda$, the activity of UX becomes zero while that of U comes back to the initial value $A_0$.

On the other hand, the growth of the activity of U could be represented by the formula:

$$ A_U = A_0 \left[ 1 - \exp (-\lambda t) \right] $$

$A_0$ and $\lambda$ are time constants; $t$ denotes the time after the separation of UX from U. Thus the decay and growth of radioactivity follow exponential laws. $A_0$ is the activity of UX immediately after its separation from U. Eqs. (3.3-1) and (3.3-2) show that after a long time ($t \to \infty$) compared to $1/\lambda$, the activity of UX becomes zero while that of U comes back to the initial value $A_0$.

In Figs. 3.1(a) and (b), the decay and growth of radioactivities of UX and U respectively are shown graphically.

The above variations of the activities of UX and U with time can be understood in the following way. As a result of the emission of radioactive radiation, the disintegrating U atoms are transformed into the atoms of a new element UX. This new element is also radioactive and is transformed into some other element by radioactive disintegration. What Crookes had measured initially as the activity of U was really the intensity
of the radioactive radiation emitted by the small amount of UX originally present in the uranium salt. The radioactive radiation emitted by \( \text{U} \) (\( \alpha \) particles) could not be detected by Crookes because of their easy absorbability. When UX was chemically separated from \( \text{U} \), the entire radioactivity was naturally observed in the precipitate containing UX. The uranium salt left over was found to be inactive. In course of time, as more and more uranium atoms disintegrated, fresh amounts of UX were formed in the uranium salt which thus regained the radioactivity that grew with time. On the other hand, due to the continual disintegration of the UX atoms emitting radioactive radiation (\( \beta \) - rays), their number gradually decreased with time. As a result the intensity of the radiation emitted by UX decreased with time.

It can be seen from Figs. 3.1(a) and (b), that the sum of the activities of UX and \( \text{U} \) is a constant (equal to \( A_0 \)) because it corresponds to uranium in equilibrium with its radioactive products.

The experiments of Crookes also showed that the activity of a radioactive substance depends on its amount. We have stated before that radioactivity is an atomic process. The atoms of a radioactive element are transformed into other atoms as a result of \( \alpha \) or \( \beta \) disintegration. According to Rutherford and Soddy, the rate of transformation of the radioactive atoms, at any instant, depends on the number of atoms present in the sample.

Consider the radioactive disintegration of the atoms of an element \( \text{P} \) into those of another element \( \text{Q} \) which can be written symbolically as:

\[
\text{P} \rightarrow \text{Q}
\]

The number of atoms of \( \text{P} \) decreases with time, while that of \( \text{Q} \) increases with time. If \( N \) be the number of atoms of \( \text{P} \) at any instant, then the rate of change \( \frac{dN}{dt} \) of \( N \) with time, which is a measure of the activity of \( \text{P} \), is proportional to \( N \) so that we can write

\[
\frac{dN}{dt} = -\lambda N
\]

or,

\[
\frac{dN}{dt} = -\lambda N
\]  ... (3.3-3)

Here \( \lambda \) is a constant, known as the disintegration constant or decay constant. The negative sign on the r.h.s. of the above equation is due to the diminution in the number of atoms \( N \) with time. Integrating Eq. (3.3-3), we get

\[
N = N_0 \exp (-\lambda t)
\]  ... (3.3-4)

At \( t = 0 \), \( N = N_0 \) which is thus equal to the number of \( \text{P} \) atoms present at the beginning of the experiment.

We can easily get the result obtained by Crookes with the help of Eq. (3.3-4). Since radioactivity \( A \) is a measure of the intensity of the radiation emitted by a radioactive substance, which depends on the rate of transformation of the radioactive atoms, we can write (except for a constant factor)

\[
A = \left| \frac{dN}{dt} \right| = \lambda N = \lambda N_0 \exp (-\lambda t) = A_0 \exp (-\lambda t)
\]

This equation is the same as Eq. (3.3-1) proposed by Crookes for decay of the activity of UX.

Suppose that the number of radioactive atoms is reduced to half in the time \( \tau \), known as the half-life of disintegration. So when \( t = \tau, N = N_0/2 \) so that we get from Eq. (3.3-4)

\[
\frac{N_0}{2} = N_0 \exp (-\lambda \tau)
\]

which gives

\[
\tau = \frac{\ln 2}{\lambda} = \frac{0.693}{\lambda}
\]  ... (3.3-5)

As \( \lambda \) increases \( \tau \) decreases and vice-versa. Eq.(3.3-4) shows that as \( \lambda \) increases, i.e., as \( \tau \) decreases, the number of radioactive atoms decreases more rapidly. Conversely if \( \lambda \) is small (\( \tau \) large), the number of radioactive atoms decreases very slowly. This is the case with uranium and thorium for which the half-lives of disintegration (\( \tau \)) are very long, so that their decay constants (\( \lambda \)) are very small.

Every radioactive substance has its own characteristic value of \( \lambda \) and \( \tau \) (see Fig. 3.2). Since the number of radioactive atoms is reduced by a factor \( \frac{1}{2} \) after a time \( \tau \), obviously the number will be reduced by the factor \( \left( \frac{1}{2} \right)^n \) or \( \frac{1}{4} \) after a time \( 2\tau \). Similarly the number will be reduced by the factor \( \frac{1}{2^n} \) or \( \frac{1}{8} \) after a time \( 3\tau \); and so on. In general, the number is reduced by a factor \( \frac{1}{2^n} \) after a time \( n\tau \), i.e., after \( n \) half-lives.

From Eq. (3.3-1), we get

\[
\ln A = \ln A_0 - \lambda t
\]  (3.3-6)
So the graph of variation of \( \ln A \) with time will be a straight line with the negative slope \(-\lambda\). It is from such graphs that \( \lambda \) and hence \( \tau \) are determined (see Fig. 3.3).

If logarithm to the base 10 is used, then the slope of the straight line will be \( \lambda \times \log_{10} e \) or 0.4343 \( \lambda \).

After a very long time \( (t \gg 1/\lambda \) or \( t \gg \tau) \) we get \( N = 0 \) and \( A = 0 \). i.e., after a very long time from the beginning of the experiment, the number of radioactive atoms \( (N) \) becomes almost zero and the radioactivity of the sample \( (A) \) disappears. In practical terms, the radioactivity becomes negligibly small after about 10 or 12 half-lives from the start of the experiment.

### 3.4 Successive disintegrations

So far we have not said anything about the radioactivity of the product \( Q \) formed in the radioactive disintegration of \( P \). \( Q \) may be stable or radioactive. If it is radioactive, then the disintegration of its atoms will lead to the formation of the atoms of another new element \( R \). If \( \lambda_1 \) and \( \lambda_2 \) are the decay constants of \( P \) and \( Q \), then we can express the successive disintegrations of \( P \) and \( Q \) as:

\[
P \rightarrow Q \rightarrow R
\]

The number of \( Q \) atoms formed per second is obviously equal to the number of \( P \) atoms disintegrating per second. If \( N_1 \) is the number of \( P \) atoms at some instant \( t \), then according to Eq. (3.3-3) the rate of disintegration of the \( P \) atoms is

\[
\frac{dN_1}{dt} = -\lambda_1 N_1
\]

Obviously the rate of production of the \( Q \) atoms is also equal to \( \lambda_1 N_1 \). Q atoms also undergo radioactive disintegration. If \( N_2 \) be the number of \( Q \) atoms present at the instant \( t \), the rate of disintegration of \( Q \) is \( \lambda_2 N_2 \). Thus the rate of change of the number of \( Q \) atoms at the instant \( t \) is

\[
\frac{dN_2}{dt} = -\lambda_2 N_2
\]

From Eq. (3.4-1), we have by integration

\[
N_1 = N_{10} \exp (-\lambda_1 t)
\]

Here \( N_{10} \) is the number of \( P \) atoms at \( t = 0 \).

To solve Eq. (3.4-2), we write

\[
N_2 = f(t) \exp (-\lambda_2 t)
\]

Differentiating we get

\[
\frac{dN_2}{dt} = \left( \frac{df}{dt} - \lambda_2 f \right) \exp (-\lambda_2 t)
\]

Substituting in Eq. (3.4-2), we get

\[
\frac{df}{dt} - \lambda_2 f = -\lambda_1 f \exp (-\lambda_2 t)
\]

Integrating gives

\[
f(t) = \lambda_1 N_{10} \exp \left[ -\lambda_1 t \right] + C
\]

where \( C \) is the integration constant. If we assume that at \( t = 0 \), the number of \( Q \) atoms \( N_2 = 0 \), then \( N_2(0) = 0 \) which gives

\[
C = \frac{\lambda_1 N_{10}}{\lambda_1 - \lambda_2}
\]

Hence we get

\[
f(t) = \frac{\lambda_1 N_{10}}{\lambda_1 - \lambda_2} \left( 1 - \exp \left[ -\lambda_1 + \lambda_2 \right] t \right)
\]

So

\[
N_2 = \frac{\lambda_1 N_{10}}{\lambda_1 - \lambda_2} \left[ \exp (-\lambda_2 t) - \exp (-\lambda_1 t) \right]
\]

Eq. (3.4-4) gives the variation of the number of \( Q \) atoms \( N_2 \) with time. From this equation, we see that \( N_2 = 0 \) at \( t = 0 \). It increases with increasing \( t \) and attains a maximum at some instant \( t_m \) which can be determined by differentiating \( N_2 \) with respect to time:

\[
\frac{dN_2}{dt} = \frac{\lambda_1 N_{10}}{\lambda_1 - \lambda_2} \left[ -\lambda_2 \exp (-\lambda_2 t) + \lambda_1 \exp (-\lambda_1 t) \right] = 0
\]

or,

\[
\lambda_2 \exp (-\lambda_2 t_m) = \lambda_1 \exp (-\lambda_1 t_m)
\]

That \( N_2 \) becomes maximum at \( t = t_m \) can easily be checked by a second differentiation of \( N_2 \). It can easily be seen that \( d^2N_2/dt^2 \) is negative at \( t = t_m \).

Sometimes it is found that the daughter product \( R \) in the radioactive disintegration of \( Q \) is also radioactive. Again the daughter product \( S \) formed from the radioactive disintegration of \( R \) may be radioactive. Such a series of successive radioactive disintegrations may be written as:

\[
P \rightarrow Q \rightarrow R \rightarrow S \rightarrow ...
\]

The last arrow after \( S \) indicates that \( S \) is also radioactive so that the radioactive chain continues after \( S \). It usually terminates after a few steps till a stable end product is formed.

If \( \lambda_1, \lambda_2, \lambda_3 \) etc. represent the decay constants and \( N_1, N_2, N_3 \) etc. are the numbers of atoms at the instant \( t \) of the elements \( P, Q, R \) etc., then we can write
\[
\begin{align*}
\frac{dN_1}{dt} &= -\lambda_1 N_1 \\
\frac{dN_2}{dt} &= \lambda_1 N_1 - \lambda_2 N_2 \\
\frac{dN_3}{dt} &= \lambda_2 N_2 - \lambda_3 N_3 \\
\frac{dN_n}{dt} &= \lambda_{n-1} N_{n-1} - \lambda_n N_n \\
\end{align*}
\]

Bateman was the first to solve this series of differential equations. A simple assumption is to suppose that at the beginning only the P atoms are present, so that \( N_2 = N_3 = N_4 = \ldots = 0 \) at \( t = 0 \).

We first consider the simple case of three successive disintegrations so that the last element S is stable:

\[
P \rightarrow Q \rightarrow R \rightarrow S \text{ (stable)}
\]

We then get, using Eq. (3.4-4)

\[
\frac{dN_3}{dt} = \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left[ \exp (-\lambda_2 t) - \exp (-\lambda_1 t) \right] - \lambda_3 N_3
\]

Multiplying by \( \exp (\lambda_3 t) \) we get

\[
\frac{dN_3}{dt} \exp (\lambda_3 t) + \lambda_3 N_3 \exp (\lambda_3 t) = \frac{d}{dt} \left[ N_3 \exp (\lambda_3 t) \right]
\]

\[
= \frac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2} \left[ \exp (-\lambda_2 t) - \exp (-\lambda_1 t) \right] \exp (\lambda_3 t)
\]

Integrating we get

\[
N_3 \exp (\lambda_3 t) = \frac{\lambda_1 \lambda_2 N_1}{\lambda_1 - \lambda_2} \left[ \frac{\exp (\lambda_2 t) - \exp (\lambda_1 t)}{\lambda_3 - \lambda_2} \right] + C
\]

C is the integration constant. Since \( N_3 = 0 \) at \( t = 0 \) we get

\[
\frac{\lambda_1 \lambda_2 N_1}{\lambda_1 - \lambda_2} \left[ \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_2} (\lambda_3 - \lambda_1) \right] + C = 0
\]

or,

\[
C = \frac{\lambda_1 \lambda_2 N_1}{(\lambda_3 - \lambda_1) (\lambda_3 - \lambda_2)}
\]

Hence we get

\[
N_3 = \lambda_1 \lambda_2 N_1 \left[ C_1 \exp (-\lambda_1 t) + C_2 \exp (-\lambda_2 t) + C_3 \exp (-\lambda_3 t) \right]
\]

...(3.4-6)

3.5 Radioactive equilibrium

The decay constants of different radioactive substances are usually different. In Figs. 3.4a and b are shown the variation in the number \( N_2 \) with time for the two cases \( \lambda_1 > \lambda_2 \) and \( \lambda_1 < \lambda_2 \) respectively, i.e. for \( \tau_1 < \tau_2 \) and \( \tau_1 > \tau_2 \) respectively.

In both cases, \( N_2 \) increases from 0 at \( t = 0 \), attains a maximum at \( t = t_m \) (see Eq. 3.4-5) and then begins to decrease. In the first case, when \( \tau_1 < \tau_2 \), the daughter element is more long-lived than the parent element. In this case, after a long time compared to \( \tau_1 \), \( N_2 \) decreases exponentially with the half-life \( \tau_2 \).

On the other hand, in the second case when \( \tau_1 > \tau_2 \), the parent element is more long-lived. After attaining the maximum, \( N_2 \) decreases exponentially with the half-life of the parent element when \( t_1 \gg \tau_2 \).

In both the cases shown in Figs. 3.4a and b, \( t_m \) is the same given by \( t_m = \ln 5 \left| \frac{\lambda_1}{\lambda_2} \right| \).

(A) Transient equilibrium: In the second case discussed above (\( \tau_1 > \tau_2 \)), transient equilibrium is established between the numbers of the
parent and product nuclei after a time long compared to the half-life of the product \((t >> \tau_2)\). Evidently, \(\lambda_2 t >> 1\) in this case, so that \(\exp(-\lambda_2 t)\) becomes negligibly small compared to \(\exp(-\lambda_1 t)\). Eq. (3.4-4) gives, in this case
\[
N_2 = \frac{\lambda_1 N_{10}}{\lambda_2 - \lambda_1} \exp(-\lambda_1 t) \quad (3.5-1)
\]
which shows that the number of atoms of the daughter element decreases with the half-life of the parent. From Eqs. (3.4-3) and (3.5-1) we then get
\[
\frac{N_2}{N_1} = \frac{\lambda_1}{\lambda_2 - \lambda_1} = \text{constant} \quad (3.5-2)
\]

Though both \(N_1\) and \(N_2\) decrease with time, the ratio remains constant. This is known as transient equilibrium.

Since the activities of the two radioactive elements are proportional to the respective numbers of their atoms, the ratio of the two activities also remains constant in time.

\[\text{(B) Secular equilibrium :}\]
When the half-life of the parent element is very long compared to that of the daughter element \((\tau_1 >> \tau_2)\) the equilibrium that is established between their numbers is known as secular equilibrium. In this case the half-life of the parent element is so long that during the time \(t << \tau_1\), the decrease in the number of the parent atoms \(N_1\) may be ignored. Obviously \(\lambda_1 t << 1\) in this case, so that \(\exp(-\lambda_1 t) = 1\). Eq. (3.4-4) then gives
\[
N_2 = \frac{\lambda_1 N_{10}}{\lambda_2} (1 - \exp(-\lambda_2 t)) \quad (3.5-3)
\]

Eq. (3.5-3) shows that the number of daughter atoms increases exponentially with time. The activity of the daughter element also increases exponentially. This was what was observed by Crookes in his experiments with UX (see § 3.3)

If the period of observation is long compared to the half-life of the daughter product \((t >> \tau_2)\), then we can write
\[
N_2 = \frac{\lambda_1 N_{10}}{\lambda_2} = N_{2w} \quad \text{(say)} \quad (3.5-4)
\]

Thus the number of daughter atoms attains a constant value \(N_{2w}\) after a time long compared to its half-time \(\tau_2\). During this time \(N_1 = N_{10}\) remains practically unchanged. Hence the ratio of the numbers of daughter and parent atoms becomes a constant:
\[
\frac{N_{2w}}{N_1} = \frac{\lambda_1}{\lambda_2} = \frac{\tau_2}{\tau_1} \quad (3.5-5)
\]

This is known as secular equilibrium. Under secular equilibrium, not only the ratio \(N_{2w}/N_1\) remains constant, but both \(N_{2w}\) and \(N_1\) also have constant values.

Eq. (3.5-5) gives
\[
\lambda_1 N_{10} = \lambda_2 N_{2w} \quad (3.5-6)
\]

This shows that though the number of daughter atoms in equilibrium \((N_{2w})\) is small compared to the number of parent atoms \(N_1 = N_{10}\), both have equal activities (see Eq. 3.3-3). The fact that the activity of UX attains a constant value when in secular equilibrium with UX is shown in Fig. 3.1(b). Eq. (3.5-3) shows that the activity of the daughter element grows with its own half-life. This is also clear from Fig. 3.1(b) where the growth of the activity \(A_U\) of uranium (which is equal to the activity of the daughter product UX in secular equilibrium) takes place with the half-life of the latter which is 24.1 days.

We have discussed about successive disintegrations in the previous section. If it is assumed that the parent element P has a half-life long compared to all its daughter products, i.e., \(\tau_1 >> \tau_2, \tau_3, \tau_4\) etc. (or \(\lambda_1 << \lambda_2, \lambda_3, \lambda_4\) etc.), then there will be secular equilibrium between all the product elements \((Q, R, S\) etc.) and the parent P after a time long compared to the half-lives of the daughter products. According to Eq. (3.5-5), the number of Q atoms attains a constant value \(N_{2w}\) after a long time \(t >> \tau_2\). So its rate of disintegration also becomes constant. Since the rate of production of the next product R is equal the rate of disintegration of Q, the former becomes constant. So after a time long compared to its half-life the number of R atoms attains a constant value. Similarly for the other successive products.

If \(N_{2w}, N_{3w}\) etc. are the numbers of atoms of the successive product elements in secular equilibrium with the parent element, then in a time \(t << \tau_1\), these numbers all become constant. Further their activities are all equal under this condition:
\[
\lambda_1 N_{10} = \lambda_2 N_{2w} = \lambda_3 N_{3w} = \lambda_4 N_{4w} = \ldots \quad (3.5-7)
\]
Secular equilibrium is maintained as long as the parent element P and the different products remain together. If any one of the daughter products is separated from the parent, then the equilibrium is disturbed. The number of the product atoms then begins to grow again and attains a saturation value. It is thus possible to separate a product element of relatively short half-life from a very long-lived parent element repeatedly.

As an example, it is possible to separate the radioactive radon gas of half-life 3.82 d from the parent element radium of half-life 1620 years every few days for its use in the treatment of cancer. It may be mentioned in this connection that in practice the equilibrium amount of the product is produced in a time of about 10 to 12 half-lives of the product.

### 3.6 Discovery of radium

Soon after the discovery of radioactivity by Becquerel, it was noticed by Madam Curie that pitch-blende, an ore of uranium, was more radioactive than pure uranium, weight for weight. She concluded that the ore must contain one or more elements which were more radioactive than pure uranium. She undertook the work of separating these unknown elements. Her husband, Pierre Currie also joined in her efforts.

The task was of stupendous magnitude since it involved the chemical separation of only a few milligramme (or even less) of the new elements whose chemical properties were totally unknown, from more than a tonne of the ore.

Undeterred by the immensity of the work she and her husband with single-minded devotion, set out to isolate the new element and began to grind sample after sample of the ore.

Since the chemical properties of the new element were not known, their only guide was the location of the radioactivity. For her work, Madam Curie had developed a new type of electrometer which was much more sensitive than the crude gold-leaf electroscope used by Becquerel. With her improved techniques she could also determine the activities of the samples quantitatively.

When the radioactivity was concentrated in a very small residue, it was clear that the new element had been found. The first such element discovered was named polonium (Z = 84) after the name of Madam Curie’s native country Poland (July, 1898).

The Curies carried out the chemical analysis of the mineral using standard methods. In the group of sulfides, insoluble in acid solution, they found polonium. They also found radioactivity in the barium group (Ba, Sr and Ca). At first they could not isolate the radioactivity from barium. When they finally succeeded in doing this by fractional crystallization, they found a new strongly radioactive substance, which they called radium (September, 1898). They were assisted in this work by the French chemist G. Bemont.

The atomic weight of radium was determined by Madam Curie to be 226 and its atomic number Z = 88. It may be worthwhile to mention that starting with more than a tonne of pitch-blende, the Curies were able to separate only about 200 mg or radium chloride. This clearly indicates the stupendous magnitude of their task.

The half-lives of polonium and radium were determined to be 139 days and 1620 years respectively. The radioactivities of these elements were found to be several orders of magnitude higher than that of uranium. In the case of radium this is about $3 \times 10^6$ times as high.

Another new element actinium (Z = 89) was discovered soon after the discovery of polonium (Po) and radium (Ra) by Andre Debierne in 1899.

It may be noted that both Po and Ra are produced as a result of successive disintegrations from uranium, which has a half-life $4.5 \times 10^9$ years. Since the half-lives of both Po and Ra are small compared to the above, these latter elements must be in secular equilibrium with the parent element uranium. For this reason, these elements are found in the uranium-bearing mineral pitch-blende. Since their radioactivities are added to that of uranium, the mineral is found to be more radioactive than pure uranium.

From Eq. (3.5-7), it is clear that the amounts of polonium and radium must be small compared to that of uranium within pitch-blende, because of the very long half-life of the latter, compared to those of the other two. As an example, we can calculate the amount of radium present in secular equilibrium with uranium.

Natural uranium has two isotopes of mass numbers 235 and 238 with the relative abundances 0.7% and 99.3% respectively. $^{226}$Ra is produced from $^{238}$U. In 1000 kg of natural uranium the number of atoms of $^{238}$U is

$$N_U = \frac{6.025 \times 10^{23}}{238} \times 10^6 \times 0.993 \times 2.514 \times 10^{27}$$

Eq. (3.5-7) gives, since $\lambda = 0.693/\tau$

$$\frac{N_{Ra}}{N_U} = \frac{\tau_{Ra}}{\tau_U}$$

or,

$$N_{Ra} = \frac{\tau_{Ra}}{\tau_U} N_U = \frac{1620}{4.5 \times 10^9} \times 2.514 \times 10^{27} = 9.05 \times 10^{20}$$

So the amount of $^{226}$Ra is

$$m_{Ra} = N_{Ra} \times \frac{226}{6.025 \times 10^{23}} = 340 \text{ mg}$$

Thus only about 340 mg of radium is present is secular equilibrium with 1000 kg of uranium. That explains why the Curies could isolate such a small amount of radium from the huge amount of pitch-blende which they had processed.

### 3.7 Radioactive series

Uranium (Z = 92) and thorium (Z = 90) are two naturally occuring elements which are radioactive. Both of these have very long half-lives.
Intensive investigations have revealed the existence of a number of short-lived daughter elements produced in the successive disintegrations of either of these elements. Evidences of their existence have been found in the minerals bearing these elements. Experiments on the growth and decay of radioactivity of these elements have established their identities and the genetic relationships between the different products.

These studies have led to the discovery of three naturally radioactive series of elements which originate from the very long lived radioactive isotopes of uranium ($^{235}$U and $^{238}$U) and from thorium ($^{232}$Th) by successive disintegrations. Each series ends in a stable isotope of lead ($Z = 82$).

Fig. 3.5, 3.6 and 3.7 show the three series known respectively as uranium-radium (or simply uranium), thorium and actinium series. The mass numbers and the atomic numbers of the daughter elements in each series are governed by displacement law discussed in § 3.2.

In the figures showing the different series, the atomic number $Z$ increases along the abscissa, while the mass number $A$ increases along the ordinate. Since $\alpha$-disintegration results in the decrease of $A$ by 4 units and of $Z$ by 2 units, this type of disintegration is indicated by a slanting arrow pointing left in the figures. On the other hand $\beta$-disintegration results in the increase of $Z$ by one unit, leaving $A$ unchanged; such a disintegration is indicated by a horizontal arrow pointing right.

For any series, $Z$ is the same for the members of the series on the same vertical line, so that they represent the isotopes of the same element of different mass numbers. Examples are RaA, RaC, RaF in Fig. 3.5, Th A, Th C in Fig. 3.6 and AcA, AcC in Fig. 3.7 which are all isotopes of polonium ($Z = 84$) of different $A$. It was F. Soddy who first discovered
(1913) the existence of isotopes during his studies on the naturally radioactive series. This was before the discovery of stable isotopes by J.J. Thomson (See Ch. II, in Vol. I).

The U–Ra series shown in Fig. 3.5 starts from the more abundant isotope $^{238}\text{U}$ of uranium which has a half-life $4.5 \times 10^9 \text{y}$. In the figure it is indicated by the symbol UI. It transforms to the element UX$_1$ by α-disintegration. Evidently $A = 234$ and $Z = 90$ for it, so that it is the isotope $^{234}\text{Th}$. It had been denoted by the symbol UX before its proper identification. Its half-life is 24.1 d and undergoes β-disintegration to produce UX$_2$ which is thus the isotope $^{234}\text{Pa}$ of the element to actinium ($Z = 91$). It is also radioactive and transforms to UX$_3$ with $Z = 92$ by α-disintegration followed by a succession of α and β disintegrations to form the stable end product $^{206}\text{Pb}$ which is the isotope $^{206}\text{Pb}$ of lead ($Z = 82$).

The thorium series starts from the isotope $^{232}\text{Th}$ of thorium ($Z = 90$). It undergoes α-disintegration having a half-life of $1.39 \times 10^{10}\text{y}$. After a number of successive α and β disintegrations it terminates at the stable end product $^{208}\text{Pb}$ (ThD) which is an isotope of lead. These are shown in Fig. 3.6.

The actinium series starts from the rarer isotope $^{235}\text{U}$ of uranium which undergoes α-disintegration with a half-life $7.1 \times 10^{10}\text{y}$. As mentioned earlier, its relative abundance in natural uranium is only 0.7%. It is also called actino-uranium. After a succession of α and β disintegrations the series terminates at the stable isotope $^{207}\text{Pb}$ (AcD), as shown in Fig. 3.7.

It may be noted that in the Th-series, the mass numbers of all the nuclides can be represented as $4n$ where $n$ is an integer. The mass numbers of the nuclides in the Th-Ra series can be represented as $4n + 2$, $n$ being an integer. Finally in the Ac-series, the mass numbers of all the nuclides, can be written $4n + 3$. For example the starting nuclei in the Th-series have the mass numbers:

- $A = 232 = 4 \times 58$ for the Th-series;
- $A = 238 = 4 \times 59 + 2$ for the U–Ra series;
- $A = 235 = 4 \times 58 + 3$ for the Ac-series.

Obviously there is no naturally radioactive series corresponding to $A = 4n + 1$. However this series has since been identified, starting with the transuranic element $^{241}\text{Pu}$ ($Z = 94$) which has half-life of 15 y. The longest lived member of the series is the $^{237}\text{Np}$ ($Z = 93$) isotope of the transuranic element neptunium. The series is called the neptunium series and is shown in Fig. 3.8. It ends in the stable isotope $^{209}\text{Bi}$ ($Z = 83$) of bismuth. Since none of the member nuclides in this series has a half-life comparable to the life of the universe, the series does not occur in nature. All its members can be produced in the laboratory by artificial means.

In Tables 3.1 to 3.4, the properties of the various members of the four series are listed.

**Table 3.1**

| Uranium-Radium Series ($4n + 2$) |
|------------------|---|---|---|---|
| **Radioactive** | **Element** | **Z** | **A** | **Symbol** | **Radiation emitted** | **Half-life** |
| Uranium I | 92 | 238 | $^{238}\text{U}$ | α, γ | $4.5 \times 10^9\text{y}$ |
| Uranium X | 90 | 234 | $^{234}\text{Th}$ | β, γ | 24.1 d |
| Uranium X | 91 | 234 | $^{234}\text{Pa}$ | β, γ | 1.18 min |
| Uranium Z | 91 | 234 | $^{234}\text{Pa}_{\alpha}$ | β, γ | 6.7 h |
| Uranium II | 92 | 234 | $^{234}\text{U}$ | α, γ | 2.5 $\times 10^5\text{y}$ |
| Ionium | 90 | 230 | $^{230}\text{Th}$ | α, γ | 8.0 $\times 10^6\text{y}$ |
| Radium | 88 | 226 | $^{226}\text{Ra}$ | α, γ | 1620 y |
| Radium Emanation | 86 | 222 | $^{222}\text{Ra}$ | α, γ | 3.82 d |
| **Radium A** | 84 | 218 | $^{218}\text{Po}$ | α (> 99%) | 3.05 min |
| Radium B | 82 | 214 | $^{214}\text{Po}$ | β (0.04%) | 26.8 min |
| Astatine-218 | 85 | 218 | $^{218}\text{Po}$ | α | 2 s |
| **Radium C** | 83 | 214 | $^{214}\text{Bi}$ | β (> 99%) | 19.7 min |
| Radium C | 84 | 214 | $^{214}\text{Bi}$ | β (> 99%) | 19.7 min |
| Radium C | 81 | 210 | $^{210}\text{Tl}$ | β, γ | 1.32 min |
| Radium D | 82 | 210 | $^{210}\text{Po}$ | β, γ | 22.3 y |
| **Radium E** | 83 | 210 | $^{210}\text{Bi}$ | β (> 99%) | 5.0 d |
| Radium F | 84 | 210 | $^{210}\text{Po}$ | α, γ | 138.3 d |
| Thallium-206 | 81 | 206 | $^{206}\text{Tl}$ | β | 4.2 min |
| Radium-G | 82 | 206 | $^{206}\text{Pb}$ | — | Stable |

The notations s, d, y denote respectively second, day and year.

** These nuclides show both α and β activities. In column 5, the branching ratios of the α and β branches are given.

*** $^{234}\text{Pa}_{\alpha}$ or UZ is an isomer of the isotope $^{234}\text{Pa}$. 

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**Radioactivity**

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Table 3.2
Thorium series (4n)

<table>
<thead>
<tr>
<th>Radioactive Element</th>
<th>Z</th>
<th>A</th>
<th>Symbol</th>
<th>Radiation emitted</th>
<th>Half-life</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thorium</td>
<td>90</td>
<td>232</td>
<td>$^{232}$Th</td>
<td>$\alpha, \gamma$</td>
<td>$1.39 \times 10^4$ y</td>
</tr>
<tr>
<td>Meso-thorium I</td>
<td>88</td>
<td>228</td>
<td>$^{228}$Ra</td>
<td>$\beta^-$</td>
<td>5.7 y</td>
</tr>
<tr>
<td>Meso-thorium II</td>
<td>89</td>
<td>228</td>
<td>$^{228}$Ac</td>
<td>$\beta^-, \gamma$</td>
<td>6.13 h</td>
</tr>
<tr>
<td>Radio-thorium</td>
<td>90</td>
<td>228</td>
<td>$^{228}$Th</td>
<td>$\alpha, \gamma$</td>
<td>1.90 d</td>
</tr>
<tr>
<td>Thorium-X</td>
<td>88</td>
<td>224</td>
<td>$^{224}$Ra</td>
<td>$\alpha, \gamma$</td>
<td>3.64 d</td>
</tr>
<tr>
<td>Thorium Emanation</td>
<td>86</td>
<td>220</td>
<td>$^{220}$Rn</td>
<td>$\alpha$</td>
<td>55.5 s</td>
</tr>
<tr>
<td>**Thorium-A</td>
<td>84</td>
<td>216</td>
<td>$^{216}$Po</td>
<td>$\alpha (&gt; 99%)$</td>
<td>0.16 s</td>
</tr>
<tr>
<td>Thorium-B</td>
<td>82</td>
<td>212</td>
<td>$^{212}$Pb</td>
<td>$\beta^-$</td>
<td>10.6 h</td>
</tr>
<tr>
<td>**Thorium-C</td>
<td>83</td>
<td>212</td>
<td>$^{212}$Bi</td>
<td>$\beta^-(66.3%)$</td>
<td>60.5 min</td>
</tr>
<tr>
<td>Thorium-C'</td>
<td>84</td>
<td>212</td>
<td>$^{212}$Po</td>
<td>$\alpha$</td>
<td>$3 \times 10^7$ s</td>
</tr>
<tr>
<td>Thorium-C''</td>
<td>81</td>
<td>208</td>
<td>$^{208}$Tl</td>
<td>$\beta^-, \gamma$</td>
<td>3.1 min</td>
</tr>
<tr>
<td>Thorium-D</td>
<td>82</td>
<td>208</td>
<td>$^{208}$Pb</td>
<td>—</td>
<td>Stable</td>
</tr>
</tbody>
</table>

3.8 Branching of radioactivity

Most natural radio-elements transform either by $\alpha$ or by $\beta^-$ disintegration. In a few cases both $\alpha$ and $\beta^-$ transformations are observed for the same isotope with definite branching ratios for the $\alpha$ and the $\beta^-$ branches. For instance, in the nuclides marked C in all the three series (RaC, Th C and Ac C) such branching of radioactivity is observed. In the U-Ra series, RaC (i.e., $^{214}$Bi) suffers $\beta^-$ disintegration more than 99% of the time while it undergoes $\alpha$-transformation 0.04% of the time, which are the corresponding branching ratios. The daughter products in the two cases are RaC' ($^{214}$Po) and RaC'' ($^{210}$Tl) respectively.
Radioactivity

Table 3.4
Neptunium series (4n + 1)

<table>
<thead>
<tr>
<th>Radioactive Element</th>
<th>Z</th>
<th>A</th>
<th>Symbol</th>
<th>Radiation emitted</th>
<th>Half-life</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Plutonium-241</strong></td>
<td>94</td>
<td>241</td>
<td>$^{241}$Pu</td>
<td>$\alpha$, $\beta^-$, $\gamma$</td>
<td>14.7 y</td>
</tr>
<tr>
<td>Uranium-237</td>
<td>92</td>
<td>237</td>
<td>$^{237}$U</td>
<td>$\alpha$, $\beta^-$, $\gamma$</td>
<td>6.75 d</td>
</tr>
<tr>
<td>Americium-241</td>
<td>95</td>
<td>241</td>
<td>$^{241}$Am</td>
<td>$\alpha$, $\gamma$</td>
<td>432 y</td>
</tr>
<tr>
<td>Neptunium-237</td>
<td>93</td>
<td>237</td>
<td>$^{237}$Np</td>
<td>$\alpha$, $\beta^-$, $\gamma$</td>
<td>$2.14 \times 10^6$ y</td>
</tr>
<tr>
<td>Protactinium-233</td>
<td>91</td>
<td>233</td>
<td>$^{233}$Pa</td>
<td>$\beta^-$, $\gamma$</td>
<td>27.0 d</td>
</tr>
<tr>
<td>Uranium-233</td>
<td>92</td>
<td>233</td>
<td>$^{233}$U</td>
<td>$\alpha$, $\gamma$</td>
<td>$1.6 \times 10^5$ y</td>
</tr>
<tr>
<td>Thorium-229</td>
<td>89</td>
<td>229</td>
<td>$^{229}$Th</td>
<td>$\alpha$, $\gamma$</td>
<td>$7.34 \times 10^3$ y</td>
</tr>
<tr>
<td>Radium-225</td>
<td>88</td>
<td>225</td>
<td>$^{225}$Ra</td>
<td>$\beta^-$, $\gamma$</td>
<td>14.8 d</td>
</tr>
<tr>
<td>Actinium-225</td>
<td>89</td>
<td>225</td>
<td>$^{225}$Ac</td>
<td>$\alpha$, $\gamma$</td>
<td>10.0 d</td>
</tr>
<tr>
<td>Protactinium-221</td>
<td>87</td>
<td>221</td>
<td>$^{221}$Pa</td>
<td>$\alpha$, $\gamma$</td>
<td>4.8 min</td>
</tr>
<tr>
<td>Astatine-217</td>
<td>85</td>
<td>217</td>
<td>$^{217}$At</td>
<td>$\alpha$, $\gamma$</td>
<td>0.032 s</td>
</tr>
<tr>
<td><strong>Bismuth-213</strong></td>
<td>83</td>
<td>213</td>
<td>$^{213}$Bi</td>
<td>$\alpha$, $\beta^-$, $\gamma$</td>
<td>45.6 min</td>
</tr>
<tr>
<td>Thallium-209</td>
<td>81</td>
<td>209</td>
<td>$^{209}$Tl</td>
<td>$\beta^-$, $\gamma$</td>
<td>2.2 min</td>
</tr>
<tr>
<td>Polonium-213</td>
<td>84</td>
<td>213</td>
<td>$^{213}$Po</td>
<td>$\alpha$</td>
<td>$4.2 \times 10^{-6}$ s</td>
</tr>
<tr>
<td>Lead-209</td>
<td>82</td>
<td>209</td>
<td>$^{209}$Pb</td>
<td>$\beta^-$</td>
<td>3.3 h</td>
</tr>
<tr>
<td>Bismuth-209</td>
<td>83</td>
<td>209</td>
<td>$^{209}$Bi</td>
<td>Stable</td>
<td></td>
</tr>
</tbody>
</table>

In the Figs. 3.5 to 3.7, the branching of radioactivity has not been shown in all cases. Thus the $\beta^-$branching of RaA or the $\alpha$-branching of RaE are not shown in Fig. 3.5, though these are listed in Table 3.1. Similarly the $\beta^-$branching of ThA or the $\alpha$-branching of Ac are not shown in Figs. 3.6 and 3.7 respectively, through these are listed in Tables 3.2 and 3.3 respectively.

If a radioactive substance exhibits both $\alpha$ and $\beta$ activities, then there are definite probabilities of decay in the two branches. Let $\lambda_\alpha$ and $\lambda_\beta$ be the decay constants for the two branches. Then the probability of that an atom of the substance will suffer $\alpha$-decay in time $dt$ is $(\lambda_\alpha dt)$ while its probability of $\beta^-$-decay is $(\lambda_\beta dt)$ in the same time interval $dt$. So the total probability of decay of the atom by both $\alpha$ and $\beta$ emissions is $(\lambda_\alpha + \lambda_\beta) dt$. If we consider $N$ atoms, the total rate of disintegration will be (see Eq. 3.3-3):

$$\frac{dN}{dt} = - (\lambda_\alpha + \lambda_\beta) N = - \lambda N \quad \text{(3.8-1)}$$

where

$$\lambda = \lambda_\alpha + \lambda_\beta \quad \text{(3.8-2)}$$

$\lambda$ is the total decay constant. If the partial half-lives of disintegration by $\alpha$ and $\beta$ emissions are $\tau_\alpha$ and $\tau_\beta$ respectively, then

$$\lambda_\alpha = \frac{\ln 2}{\tau_\alpha} \quad \text{and} \quad \lambda_\beta = \frac{\ln 2}{\tau_\beta}$$

If $\tau$ is the mean half-life of the substance, then we can write

$$\lambda = \frac{\ln 2}{\tau}$$

So we get from Eq. (3.8-2)

$$\frac{\ln 2}{\tau} = \frac{\ln 2}{\tau_\alpha} + \frac{\ln 2}{\tau_\beta}$$

or

$$\frac{1}{\tau} = \frac{1}{\tau_\alpha} + \frac{1}{\tau_\beta}$$

The ratios $\lambda_\alpha/\lambda$ and $\lambda_\beta/\lambda$ give the branching ratios for the two branches.

3.9 Radon gas

In each naturally radioactive series, a radioactive isotope of the element $Z = 86$ is produced which is gaseous at ordinary room temperature. Actually it is the heaviest known inert gas. In the early days, different names had been proposed for it. In the case of the U-Ra series the name was radium-emanation ($^{222}$Em). In the cases of the thorium and actinium series, the names were thorium-emanation ($^{232}$Em) and actinium-emanation ($^{226}$Em) respectively. Later, the common name radon (Rn) was proposed for it. This element is similar to the inert gases neon, argon, krypton etc. and falls in the last column of the periodic table. It has the outermost 6p shell completely filled with 6 electrons.

In the early years of radioactivity, the scientists working in the different laboratories used to face problems with this gas whose existence was unknown at the time. During chemical processing of the naturally radioactive substances, this unknown gas would escape from the samples and would spread to different places in the laboratory. Due to their relatively short half-life, they would transform into their daughter products within a short time. These daughter products being solid would be deposited at different places (e.g., tables, walls etc.) in the laboratory. Since these active-deposits were $\alpha$-active the various objects in the laboratory used to exhibit radioactivity. The workers in the laboratories were greatly puzzled at these happenings. However Rutherford and R.B. Owens realized that the real reason behind these mysterious happenings was the emission of the above mentioned radioactive gases. Rutherford and Soddy were able to liquify this gas by cooling it to $-150^\circ$C which conclusively established its true nature and resolved the above mentioned puzzle. The radon gas is usually separated from the parent element (and other products) by means of an air current sent through a glass tube.
3.10 Unit of radioactivity

The unit of radioactivity was originally defined as the number of disintegrations which take place in 1 gm of $^{226}$Ra by $\alpha$-particle emission in one second. The unit has been given the name Curie. This number has been measured very carefully by various workers and has a value close to $3.7 \times 10^{10}$ disintegrations per second. However, these measurements did not always yield consistent results. So the old definition was replaced by a new definition in 1950. According to this

$$1\text{ curie} = 3.70 \times 10^{10} \text{ disintegrations/s}$$

If the number of atoms in a radioactive substance disintegrates at the above rate, it is said to have a radioactivity of 1 curie (Ci).

One thousandth part and one millionth part of a curie are known as a millicurie (mCi) and a microcurie (μCi) respectively.

Above rate, it is said to have a radioactivity of 1 curie (Ci).

A new definition in 1950. According to this

The unit of radioactivity was originally defined as the number of disintegrations per second. However, these measurements did not always yield consistent results. So the old definition was replaced by i

$$\lambda Rn \text{ N}_{Rn} = 1 \text{ Ci} = 3.7 \times 10^{10} \text{ disintegrations/s}$$

Here $\lambda_{Rn}$ is the decay constant of $^{226}$Rn, $\tau_{Rn}$ being its half-life: $\tau_{Rn} = 3.82 \text{ d}$. We then get the number of $^{226}$Rn atoms

$$N_{Rn} = \frac{3.7 \times 10^{10} \times 3.82}{0.693} \times 24 \times 3600 = 1.762 \times 10^{16}$$

Then the amount of $^{226}$Rn is

$$m \text{ (Rn)} = \frac{1.762 \times 10^{16} \times 222}{6.025 \times 10^{23}} = 6.49 \times 10^{-6} \text{ g}$$

The volume of this amount of $^{226}$Rn gas at S.T.P. is

$$V_{Rn} = \frac{6.49 \times 10^{-6} \times 22417}{222} = 6.55 \times 10^{-4} \text{ cm}^3 = 6.55 \times 10^{-10} \text{ m}^3$$

Similarly for $^{238}$U we get ($\tau_{U} = 4.5 \times 10^9 \text{ y}$)

$$N_{U} = \frac{3.7 \times 10^{10}}{0.693} \times 4.5 \times 10^9 \times 365 \times 24 \times 3600 = 7.58 \times 10^{27}$$

So the amount of $^{238}$U is

$$m \text{ (U)} = \frac{7.58 \times 10^{27} \times 238}{6.025 \times 10^{23}} = 2994 \text{ kg}$$

3.11 Mean life of a radioactive substance

Radioactive disintegration is an entirely random process. It is not possible to predict in advance as to when a particular atom will disintegrate. If we confine our attention on a particular radioactive atom it may disintegrate right at the moment we are looking at it; or it may disintegrate a long time afterwards. This statement holds for each and every atom in the specimen. However, if we consider the very large number of atoms present in the specimen, then the atoms in it will be found to disintegrate at a certain definite rate, subject to some fluctuations. The rate of disintegration is determined by the decay constant $\lambda$ which has a definite value for a given radioactive substance. It has different values for different radioactive substances.

If we start from a particular instant ($t = 0$), then the different atoms will disintegrate at different instants of time after the start. So different atoms have different life-times for their existence. The atoms which disintegrate within the interval $t$ to $t + dt$ from the start may be regarded as having a mean life of $t$. Suppose their number is $dN$. Then from Eq. (3.3-4) we have

$$dN = -\lambda N dt = -\lambda N_0 \exp (-\lambda t) dt$$

The mean life $\tau_m$ of all the atoms in the given substance will then be

$$\tau_m = \int_0^\infty \frac{t dN}{dN} = \frac{-\lambda}{\lambda} \int_0^\infty N_0 \exp (-\lambda t) dt = \frac{1}{\lambda}$$

It should be noted that since the time of disintegration of the atoms extends from 0 to infinity, the mean life is not really of much physical significance. In practical terms, the decay constant $\lambda$ and the half-life of disintegration $\tau$ are more significant quantities.

Using Eq. (3.3-5), we get the following relationship between $\tau_m$ and $\tau$:

$$\tau_m = \frac{\tau}{\ln 2} \quad \text{or} \quad \tau = 0.693 \tau_m$$

3.12 Measurement of decay constant

Various methods have been developed for the measurement of the decay constants of radioactive substances. We shall discuss some of the more important methods below.

(i) For half-lives, extending from a few seconds upto about 10-15 years, it is possible to measure the decay constant (or the half-life) by following the decrease of the intensity of the radiation emitted by a radioactive substance (activity) with time. The plot of the natural logarithm of the activity with time then gives a straight line with a slope ($-\lambda$). The method was discussed in § 3.3 (see Eq. 3.3-6).

This method is known as the direct method. If the activity of the sample is very feeble, then the accuracy of the method is not very good.

The intensity of the radiation is usually measured with a suitable particle (or photon) counting device, e.g., a Geiger-Müller counter or a scintillation counter to be described in Ch.VII.
Radioactivity

In some cases, the recoiling short half-lived decay product is deposited at one point on a rotating disc, along the circumference of which are placed a succession of suitable detectors, placed at regular intervals, as shown in Fig. 3.10. If the disc is rotated at a uniform speed, the deposited radioactive substance passes by the window of the successive detectors at regular intervals of time.

The intensity of the radiation emitted by the radioactive substance decreases as the deposited source goes from one detector to the next.

Half-life of $^{216}$Po (ThA) has been measured by this rotating disc method. The half-life is 0.158 s.

For the measurement of extremely short half-life (10⁻³ s or less), the delayed coincidence method is most suitable. In this method, there are two detectors for detecting two correlated events occurring within a very short time interval of each other. As an example, in the decay $^{212}$Bi →...
\[ A = A_1 + A_2 = A_{10} \exp(-\lambda_1 t) + A_{20} \exp(-\lambda_2 t) \] \hspace{1cm} (3.13-1)

where \( A_{10} \) and \( A_{20} \) are the activities at \( t = 0 \). At the beginning when \( t \ll \tau_2, \lambda_2 \) is small so that \( \exp(-\lambda_2 t) \approx 1 \). Hence we can write for small \( t \):

\[ A = A_{10} \exp(-\lambda_1 t) + A_{20} \]

or,

\[ \ln (A - A_{20}) = \ln A_{10} - \lambda_1 t \] \hspace{1cm} (3.13-2)

Thus the graph of the activity in a semilog plot for large \( t \) is a straight line with a negative slope \( -\lambda_1 \). This is shown by the tail portion of the decay curve (portion CD). If this portion is extrapolated back to zero time, we get \( A_{20} \), the initial activity of the element 2.

If now \( A_{20} \) is subtracted from the total activity \( A \) for small \( t \), then Eq. (3.13-2) shows that the variation of \( \ln (A - A_{20}) \) with \( t \) is a straight line with a negative slope \( -\lambda_1 \). When extrapolated back to \( t = 0 \), this yields the value of \( A_{10} \), the initial activity of the element 1.

Thus the resultant complex decay curve is separated into two decay curves for the two components by this 'peeling-off' method and the corresponding half-lives \( \tau_1 \) and \( \tau_2 \) can be determined.

The method can be extended to the case of more than two components if the half-lives of the components are fairly different.

If the half-lives of the components are comparable the analysis is more difficult.

3.14 Age of minerals and rocks

There are strong evidences, astronomical and geophysical, that the elements present in the universe were created about 5.5 billion \( (10^9) \) years ago, while the solidification of the earth's crust took place about 3.5 billion years ago. Studies on the radioactivity of very long-lived natural radioelements e.g., uranium and thorium, have confirmed these. If the half-lives of these elements were much shorter than the time which has elapsed since the creation of these elements in the universe, they would not have been able to survive and no trace of their existence would be found anywhere in the universe. Comparison of the half-lives of uranium and thorium quoted earlier with the age of the universe \( (\sim 10^{10} \text{y}) \) show that they are of the same order of magnitude.

It is possible to determine the geological ages of the minerals and rocks containing the above elements from the study of their radioactivity. We shall briefly discuss some of these methods below.

(i) Lead method: We saw in § 3.7 that the different natural radioactive series start from the very long-lived radioisotopes \(^{238}\text{U}, \(^{232}\text{Th}\) or \(^{235}\text{U}\) and after suffering a number of successive \( \alpha \) and \( \beta \) disintegrations the radioactive atoms transform into the stable end products which are the isotopes of lead.

In the case of the U-Ra series, the successive, disintegrations result in the formation of one atom of the stable isotope \(^{206}\text{Pb}\) of lead from each
atom of $^{238}\text{U}$. If the time which has elapsed since the solidification of the minerals bearing uranium in the earth’s crust until the present date is $t$, and if $N_{U_0}$ was the number of atoms of $^{238}\text{U}$ which were present initially in a given amount of the mineral, then the number of $^{238}\text{U}$ atoms which are present in the sample at present is

$$N_U = N_{U_0} \exp (-\lambda_U t) \quad \text{(3.14-1)}$$

So the number of $^{238}\text{U}$ atoms which have undergone radioactive transformation during this time and which must be equal to the number of $^{206}\text{Pb}$ atoms formed due to these transformations is

$$N_{Pb} = N_{U_0} - N_U = N_U [\exp (\lambda_U t) - 1] \quad \text{(3.14-2)}$$

From the above equation we get

$$t = \frac{1}{\lambda_U} \ln \frac{N_{Pb} + N_U}{N_U} \quad \text{(3.14-3)}$$

$N_{Pb}$ and $N_U$ can be determined by chemical and mass spectroscopic methods so that the geological age of the mineral $t$ can be determined using Eq. (3.14-3).

Similarly, the age of a thorium bearing mineral can be found by determining the ratio of $^{208}\text{Pb}$ to $^{232}\text{Th}$.

The above method is based on the assumption that the mineral under study does not contain any lead other than the radiogenic lead isotope ($^{206}\text{Pb}$) originating due to the radioactive disintegration of $^{238}\text{U}$. Further it is also assumed that the entire amount of lead thus produced has been retained within the mineral. These assumptions seem to be reasonable. Even so, some loss of lead due to leaching cannot be ruled out. The above method is known as the lead method.

(ii) Helium method: The $\alpha$-particles emitted during the successive disintegrations starting from $^{238}\text{U}$ turn into neutral helium atoms by acquiring orbital electrons which accumulate as helium gas within the mineral containing uranium. During the long period since the solidification of the mineral, measurable quantity of helium gas is produced. If its amount is measured, then the age of the mineral can be determined.

An inspection of Fig. 3.5 shows that transformation of $^{238}\text{U}$ into $^{206}\text{Pb}$ takes place by successive disintegrations during which 8 $\alpha$-particles are emitted. Hence the number of helium atoms formed is

$$N_{He} = 8 \left[ N_{U_0} - N_U \right] = 8 N_U [\exp (\lambda_U t) - 1] \quad \text{(3.14-4)}$$

This gives

$$t = \frac{1}{\lambda_U} \ln \frac{8 N_U + N_{He}}{8 N_U} \quad \text{(3.14-5)}$$

Thus by measuring the amount of the helium gas accumulated within a given sample containing a known amount of $^{238}\text{U}$, it is possible to measure the life $t$ of the mineral.

Radioactivity

It may be noted that helium gas is produced due to the successive $\alpha$-disintegrations of both the isotopes $^{238}\text{U}$ and $^{235}\text{U}$ of uranium. The latter emits 7 $\alpha$-particles before an atom of $^{235}\text{U}$ is transformed into the stable $^{207}\text{Pb}$ isotope so that 7 helium atoms are produced for each atom of $^{235}\text{U}$ transformed. When this is taken into account, the number of atoms of helium comes out to be

$$N_{He} = 8 N_U [\exp (\lambda_U t) - 1] + 7 N_U' [\exp (\lambda'_U t) - 1] \quad \text{(3.14-6)}$$

Here $\lambda_U$ and $\lambda'_U$ are the decay constants of $^{238}\text{U}$ and $^{235}\text{U}$ respectively and $N_U$ and $N'_U$ are the numbers of their atoms in the given amount of the material.

The above method presumes that all the helium, accumulated within the sample since the beginning of its formation, is retained by it. Besides, no helium gas from any other source is present in the mineral. Both these assumptions are faulty. Some helium gas is likely to diffuse in or out of the porous structures of the rocks.

(iii) Isotopic abundance method: In this method the ratio of the relative abundances of the lead isotopes $^{206}\text{Pb}$ and $^{207}\text{Pb}$ of radiogenic origin are measured by mass spectroscopic method, from which the age of the mineral can be estimated. This is probably the best method for estimating the age of minerals. It is less sensitive to chemical and mechanical losses of lead from the mineral. Absence of any $^{204}\text{Pb}$ isotope ensures radiogenic origin of the two lead isotopes $^{206}\text{Pb}$ and $^{207}\text{Pb}$ in the mineral.

(iv) Potassium method: Naturally occurring potassium ($Z = 19$) has a long-lived radioactive isotope $^{40}\text{K}$ which suffers $\beta^+$ disintegration and transforms into the stable isotope $^{40}\text{Ar}$ of argon ($Z = 18$). Its half-life is $1.3 \times 10^9$ y. Natural argon has three stable isotopes $^{36}\text{Ar}$, $^{38}\text{Ar}$, and $^{40}\text{Ar}$ with the relative abundances $0.337\%$, $0.063\%$ and $99.6\%$ respectively. So the normal ratio of $^{40}\text{Ar}$ to $^{36}\text{Ar}$ is known.

If this ratio (measured by mass spectroscopic method) is found to be different in the case of argon gas derived from a potassium bearing mineral, then from a comparison of the measured ratio with the normally expected ratio, it is possible to estimate the age of the mineral.

Apart from uranium and thorium, the existence of certain very long-lived isotopes of some elements are known. A list of these is given in Table 3.5.

The ages of the minerals bearing some of these elements have been estimated, using the radioactivity of their isotopes listed in the table.

In some cases, because of their extremely long half-lives, the radioactivities of the isotopes are so feeble that it is very difficult to detect them. So it may be possible that in future some of the isotopes known to be stable may actually be found to be radioactive with extremely long half-lives.
Table 3.5

<table>
<thead>
<tr>
<th>Atomic number</th>
<th>Isotope</th>
<th>Nature of radioactivity</th>
<th>Half-life (years)</th>
</tr>
</thead>
<tbody>
<tr>
<td>19</td>
<td>⁴⁰K</td>
<td>E.C., β⁺, β⁻</td>
<td>1.3 x 10⁹</td>
</tr>
<tr>
<td>23</td>
<td>⁵⁰V</td>
<td>E.C.</td>
<td>4 x 10¹⁶</td>
</tr>
<tr>
<td>37</td>
<td>⁸⁷Rb</td>
<td>β⁻</td>
<td>5 x 10¹⁰</td>
</tr>
<tr>
<td>49</td>
<td>¹¹⁵In</td>
<td>β⁻</td>
<td>5 x 10¹⁴</td>
</tr>
<tr>
<td>57</td>
<td>¹³⁴La</td>
<td>E.C., β⁻</td>
<td>10¹¹</td>
</tr>
<tr>
<td>58</td>
<td>¹⁴²Ce</td>
<td>α</td>
<td>&gt; 5 x 10¹⁶</td>
</tr>
<tr>
<td>60</td>
<td>¹⁴⁰Nd</td>
<td>α</td>
<td>2.1 x 10¹⁵</td>
</tr>
<tr>
<td>62</td>
<td>¹⁴⁷Sm</td>
<td>α</td>
<td>1.1 x 10¹¹</td>
</tr>
<tr>
<td>71</td>
<td>¹⁷⁶Lu</td>
<td>E.C., β⁻</td>
<td>2.9 x 10¹⁰</td>
</tr>
<tr>
<td>75</td>
<td>¹⁸⁷Re</td>
<td>β⁻</td>
<td>5 x 10¹⁰</td>
</tr>
<tr>
<td>78</td>
<td>¹⁹⁹Pt</td>
<td>α</td>
<td>6 x 10¹¹</td>
</tr>
<tr>
<td>83</td>
<td>²⁰⁹Bi</td>
<td>α</td>
<td>≥ 2 x 10⁸</td>
</tr>
</tbody>
</table>

There is another class of radioactive isotopes occurring in nature which have relatively much shorter half-lives. These include ³H (τ = 12.4 y), ¹⁴C (τ = 5330 y) etc. However they do not belong to the same class as those listed in Table 3.5. Whereas, the latter were produced at the beginning of creation and have survived the long span of life of the universe, these shorter half-lived isotopes are being continually created by the action of the high energy particles present in the cosmic rays (see Ch.X) on the stable nuclei of various elements present in the atmosphere or in the earth's crust. In course of time an equilibrium is established between the rates of their production and disintegration so that an equilibrium amount is always present in nature.

The radioactive property of ¹⁴C isotope has been utilized in age determination of archaeological objects upto about 25,000 years. He can also be used in dating, though its application is limited by its relatively short half-life and low abundance.

The two isotopes ¹⁴C and ⁴⁰K are of considerable biological interest since they are responsible for the appreciable radiation levels observed in the bodies of all living organisms within which they are always present. For example, though ⁴⁰K is present to the extent of only 0.0119% in natural potassium, the gamma radiation emitted by it is readily detectable in the human body. Several hundred counts per minute can be detected in the whole body by a gamma-ray counter (see Ch. XV).

Radioactivity

1. The variation of the rate of β-particle emission from a radioactive sample with time is given in the following table:

<table>
<thead>
<tr>
<th>Time (hrs.)</th>
<th>Rate of emission</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2169</td>
</tr>
<tr>
<td>10</td>
<td>1458</td>
</tr>
<tr>
<td>20</td>
<td>1007</td>
</tr>
<tr>
<td>35</td>
<td>609</td>
</tr>
<tr>
<td>50</td>
<td>346</td>
</tr>
<tr>
<td>65</td>
<td>212</td>
</tr>
<tr>
<td>80</td>
<td>225</td>
</tr>
<tr>
<td>95</td>
<td>71</td>
</tr>
</tbody>
</table>

Plot the graph of the logarithm to the base 10 of the rate of β-emission with time and determine the half-life of the sample. What is its decay constant?

2. A radioactive element has a half-life of 25 hours. After what time will 1/8th of initial number of its atoms disintegrate? After what time will 1/32 of the initial number remain unchanged? (4 h 49 min; 125 h)

3. 3.25 pg of radon gas (²²²Rn) is separated from a certain amount of radium (²²⁶Ra) in secular equilibrium with its products. If the half-lives of radon and radium are 3.84 d and 1620 y respectively, what is the amount of radium? (0.51 g)

4. Prove that in the successive radioactive disintegration X → Y → Z, the time t_m given by Eq.(3.4-5) is the time in which the number of Y atoms become maximum.

If t₁ = 10 h and t₂ = 1.3 h, calculate t_m. What will be the value of t_m if the half-lives are t₁ = 1.3 h and t₂ = 10 h? (4.4 h)

5. Potassium is present in the human body to the extent of about 0.35% of the body weight. Calculate the total activity due to ⁴⁰K in a man weighing 70 kg, assuming the relative abundance of ⁴⁰K in potassium to be 0.012%.

(0.20 Ci)

6. In a uranium ore, the amount of ²⁰⁶Pb is found to be 0.5 kg per kilogram of U. Assuming that all of ²⁰⁶Pb has been produced from ²³⁵U by radioactive disintegrations calculate the life of the ore.

(2.96 x 10⁹ y)

7. A certain uranium bearing mineral yields 2.8 x 10⁷ kg of ²³⁸U and 10 x 10⁻¹⁷ m³ of helium gas at S.T.P. for each kilogram of the ore. What is
the age of the mineral? What is the error in neglecting the production of helium in the disintegration of $^{235}\text{U}$?

(1.333 x $10^9$ y)

8. Calculate the recoil velocity of a $^{214}\text{Po}$ atom produced by the $\beta$-decay of $^{214}\text{Bi}$, when the $\beta$-particle is emitted with the maximum energy, which is 3.28 MeV.

(5.64 x $10^3$ m/s)

9. The radioactivity of a sample containing two radioactive substances of different half-lives is measured with the help of a Geiger-Müller counter. The variation of the counting rate (which is a measure of the activity) after correcting for the cosmic ray background with time, is given below:

<table>
<thead>
<tr>
<th>Time (h)</th>
<th>0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting rate</td>
<td>5056</td>
<td>2576</td>
<td>1413</td>
<td>842</td>
<td>568</td>
<td>425</td>
<td>310</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time (h)</th>
<th>4.0</th>
<th>6.0</th>
<th>8.0</th>
<th>10.0</th>
<th>12.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Counting rate</td>
<td>280</td>
<td>211</td>
<td>167</td>
<td>131</td>
<td>105</td>
</tr>
</tbody>
</table>

Make a semi-log plot of the above data and find the half-lives of the two components. (0.48 h; 6.1 h)

10. $^{231}\text{Pa}$ is in secular equilibrium with the rarer isotope $^{235}\text{U}$ (0.7%) of uranium. A measurement of the ratio of the atoms of natural uranium to $^{231}\text{Pa}$ gives a value $2.89 \times 10^6$. Calculate the half-life of $^{235}\text{U}$, assuming that of $^{231}\text{Pa}$ to be $3.3 \times 10^4$ y. (6.6 x $10^8$ y).

11. Calculate the age of a mineral in which the ratio of $^{206}\text{Pb}$ to $^{207}\text{Pb}$ is found to be 14.0 and which is essentially free of $^{204}\text{Pb}$.

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**Alpha Particles and Alpha Radioactivity**

4.1 Determination of $q/M$ of $\alpha$-particles

We saw in the previous chapter that Rutherford discovered $\alpha$-particles as one of the constituent radiations emitted by naturally radioactive elements and identified them as the nuclei of $^4\text{He}$ ($Z=2$) atoms. Two other constituents of these radiations are the $\beta$-particles (high energy electrons) and $\gamma$-rays (very short wavelength electromagnetic radiation). In the early years of radioactivity, these different radiations were distinguished by the differences in their absorbability in matter (see § 3.1). Further, their behaviour in a magnetic field was also found to be different as shown in Fig. 4.1. The $\alpha$-rays are only slightly deflected by a magnetic field, while the $\beta$-rays are much more strongly deflected, $\gamma$-rays are not deflected at all showing that they are electrically neutral. The deflections of the $\alpha$ and $\beta$-rays show that they are electrically charged particles and carry opposite charges. The much stronger deflection of the $\beta$-particles show that they are much lighter than the $\alpha$-particles.

![Fig. 4.1. Paths of $\alpha$, $\beta$ and $\gamma$ rays in a magnetic field.](image1)

![Fig. 4.2. Experimental arrangement for measuring $q/M$ of the $\alpha$-particles.](image2)

Rutherford and Robinson were the first to measure the specific charge (the ratio $q/M$ of charge to mass) of the $\alpha$-particles by an
Alpha Particles and Alpha Radioactivity

the distance between A and P, then the deflection of the α-particles on the plate P becomes

\[ d_\alpha = \frac{Xq}{M} \frac{L}{2v} = \frac{Xq LL}{2Mv^2} \]

or,

\[ \frac{q}{Mv^2} = \frac{2d_\alpha}{XL} \quad (4.1-1) \]

Here \( \tau = L/v \) is the time taken by the α-particles to travel from the slit to the plate P.

Eq. (4.1-1) involves the velocity v of the α-particles. Rutherford and Robinson performed a magnetic deflection experiment to eliminate v.

The principle of the measurement is shown in Fig. 4.3. The source S of α-particles was a thin platinum wire which was placed inside radon gas for a few days. As a result, the wire was coated with the active deposits RaA, RaB and RaC from the radon gas (see § 3.9). The source mainly gives off α-particles from RaC. The α-particles pass through the slit A parallel to S and fall on the photographic plate P. The apparatus is evacuated and placed in a strong uniform magnetic field applied perpendicular to the plane of the paper. The paths of the α-particles are arcs of circles which pass through the slit A and intersect the plate P at some point F above or below the straight line SA intersecting P at O. When the magnetic field is reversed, the α-particles hit the plate at the symmetrically opposite point on the other side of SAO. As in the previous case, we thus obtain two dark lines on P.

If the magnetic induction field is B, then the magnetic force is \( Bqv \) so that we can write

\[ Bqv = \frac{Mv^2}{R} \]

Here R is the radius of the circular arc described by the α-particles. We thus get

\[ \frac{q}{M} = \frac{1}{BR} \quad (4.1-2) \]

From Fig. 4.4, we see that if the vertical height of the highest point M of the circular arc above the slit A be s, then

\[ (2R-s)s = \left(\frac{l}{2}\right)^2 \]

Fig. 4.3. Determination of \( q/Av \) of the α-particles. The magnetic field is perpendicular to the plane of the paper.

Fig. 4.4. Determination of the radius of the path of an α-particle in a magnetic field.
or, \[ 2R \frac{s}{d_m} = \frac{p^2}{4} \] (\because s < 2R) \hspace{1cm} \text{(4.1-3)}

Here \( d_m \) is the distance from \( S \) to \( A \). Since the magnetic field acts on the whole apparatus from \( S \) to \( P \), the \( \alpha \)-particles are acted upon by the field even after their emergence through the slit. From the figure, we then have

\[ (2R - s - d_m)s + d_m = \left( \frac{1}{2} + L \right)^2 \]

Here \( d_m \) is the vertical deflection of the \( \alpha \)-particles from the line \( SA \) when it reaches the plate \( P \). \( L \) is the distance from the slit to the plate. Since \( s + d_m < 2R \), we can write

\[ 2R(s + d_m) = \frac{p^2}{4} + L(L + L^2) \]

From Eqs. (4.1-3) and (4.1-4), we have

\[ 2Rd_m = LL + L^2 \]

or, \[ R = \frac{LL + L^2}{2d_m} \]

Hence from Eq. (4.1-2) we get

\[ \frac{q}{Mv} = 2 \frac{d_m}{B(L(L + L^2))} \] \hspace{1cm} \text{(4.1-5)}

From Eqs. (4.1-1) and (4.1-5) we then get finally after eliminating \( v^2 \),

\[ \frac{q}{M} = 2 \frac{L^2 d_m^2}{B^2 L(L + L^2) d_m} \] \hspace{1cm} \text{(4.1-6)}

\( q/M \) for the \( \alpha \)-particles can be determined using Eq. (4.1-6).

Rutherford and Robinson found

\[ \frac{q}{M} = 4820 \times 10^7 \text{ coulombs/kg} \]

The above experiment also gave the velocity of the \( \alpha \)-particles which was found to be extremely high, that from RaC' being \( 1.92 \times 10^7 \text{ m/s} \) which corresponds to an energy of 7.68 MeV.

### 4.2. Determination of the charge and mass of the \( \alpha \)-particles

To establish the identity of the \( \alpha \)-particles, it is necessary to know their charge and mass separately. If the charge \( q \) is determined experimentally, then from the value of the specific charge determined above, it is possible to find the mass \( M \).

Rutherford and H. Geiger were the first to determine the charge of the \( \alpha \)-particles. They first measured the number of \( \alpha \)-particles emitted by a radioactive source per second. Then they measured the total quantity of charge carried by them. From this, the charge of each \( \alpha \)-particle could be determined.

In order to determine the rate of emission of the \( \alpha \)-particles from a radioactive substance, it is necessary to count the particles individually.

Various methods have been developed for the detection and counting of the individual ionizing particles.

When Rutherford and Geiger performed their experiments towards the beginning of the present century (1905), they had mainly used two different methods. Though these methods were not as sensitive as the present day methods for particle counting, they were able to make fairly accurate measurements with their apparatus.

In one of the methods, they used a screen coated with ZnS. The \( \alpha \)-particles produced momentary flashes of light known as scintillations on such a screen. So if an \( \alpha \)-emitting substance is kept in front of such a screen within a dark room the screen would become faintly illuminated. Since a large number of \( \alpha \)-particles is simultaneously incident on the screen, each particle producing a momentary flash of light, the total effect is to make the screen appear uniformly illuminated. So in order to detect the individual \( \alpha \)-particles by the scintillations they produce, it is necessary to use a very small screen. The screen coated with ZnS is mounted in front of the objective of a microscope and viewed through the latter. Because of the extremely small area of the screen visible through the objective lens, it is possible to observe the scintillations produced by the individual \( \alpha \)-particles falling on it from a source placed just behind it. Rutherford and Geiger used such an arrangement to determine the number of \( \alpha \)-particles emitted by the source per second. Their experimental arrangement is shown in Fig. 4.5.

Let \( A \) be the area of the screen \( P \) seen through the microscope \( M \), \( r \) the distance of the screen from the source \( S \) and \( N \) the number of \( \alpha \)-particles emitted by the source per second. Then the number of \( \alpha \)-particles falling on the screen per second is

\[ n = \frac{N \rho A}{4 \pi r^2} \]

This gives \[ N = \frac{4 \pi r^2 n}{A} \] \hspace{1cm} \text{(4.2-1)}

In the second method for determining the number of \( \alpha \)-particles emitted by a source, Rutherford and Geiger used a particle counting device invented by Geiger and known as a Geiger counter. Improved version of this apparatus will be described in Ch. VII. As in an ionization chamber, the counter consisted of a cylindrical metallic cathode tube along the axis of which was stretched a thin metallic wire, the whole arrangement being enclosed in an air-tight chamber, kept under very low pressure. One end of the chamber was covered by a thin mica foil, serving...
as a window through which the α-particles could enter the counter. A potential difference of about 1000 volts was applied between the anode and the cathode. When a high energy α-particle travels through the low pressure air inside the counter, it ionizes the air molecules and a large number of positive and negative ions are produced in the chamber. These ions again produce secondary ion pairs by collision with the air molecules. All these ions are attracted by the oppositely charged electrodes, as a result of which a momentary spark is produced. In the experiment of Rutherford and Geiger, the axial anode wire was connected to a quadrant electrometer which showed a sudden deflection when a large number of ions produced in the counter gas fell on the anode. Thus the entry of each α-particle was indicated by the deflection of the electrometer needle. By counting the number of such deflections, the rate of entry of the α-particles into the counter through the window could be determined.

Knowing the area of the window and the distance of the α-source from the latter, the rate of emission of the α-particles by the source could then be determined with the help of a formula similar to Eq. (4.2-1).

Later Rutherford and Geiger used a string electrometer of small time-period in place of the quadrant electrometer (1912). By this replacement, the counter became more sensitive and they were able to record up to 1000 α-particles per minute entering the counter.

After determining the rate of emission of α-particles from a given amount of the radioactive source, Rutherford and Geiger proceeded to measure the total amount of charge carried by these α-particles. Their experimental arrangement is shown in Fig. 4.6.

S is a shallow dish, containing a known quantity of radium covered by a thin aluminium foil which allowed the α-particles to pass through it, but kept back the recoil daughter atoms. The α-particles coming from S then passed through a thin diaphragm D of known area and were incident on the collector plate P covered by a thin aluminium foil. The collector plate was connected to an electrometer which measured the charge collected by it.

The whole apparatus was kept under high vacuum and was subjected to a strong magnetic field. This field deflected the β-particles, also emitted from the source S and prevented them from reaching the collector plate P. It also kept back the electrons ejected from P due to the impact of the incident α-particles on it (β-rays).
attached to B which was connected to a mercury reservoir through a rubber tube. By raising or lowering this reservoir, the level of mercury in the lower part of B could be adjusted as desired.

At the beginning of the experiment, the air within B and C was pumped out. The α-particles emitted by the radon gas in A passed through the wall of the latter and entered the empty space within B. Because of the very low gas pressure within B, they hit the inner wall of B and captured electrons to form neutral helium atoms, which gradually spread into the empty space within B as helium gas. After a few days, sufficient helium gas collected within B. By adjusting the height of the mercury reservoir connected to B, this helium gas was then compressed into the capillary C.

A high voltage was then applied between the electrodes P and Q in C and a luminous discharge was produced between them. The light thus emitted was then analyzed with the help of a spectrometer.

Rutherford and Royds found the presence of helium spectral lines in this light, which proved conclusively that the α-particles were nothing but doubly charged helium ions (He⁴⁺).

4.4. Determination of the velocity of α-particles

We have seen that the α-particles are emitted with very high velocity from radioactive substances. As already stated (§ 4.1) the velocity of the RaCα α-particles is 1.92 × 10⁷ m/s which is about 1/16 the velocity of light. We also saw that Rutherford and Robinson’s experiment on the determination of $qM$ of the particles gave their velocity.

Later S. Rosenblum designed a magnetic spectrograph to measure the α-particle velocities from different sources very accurately. The instrument is similar to the one used for the measurement of the β-ray spectra and will be described in Ch. V in detail.

In this instrument, a very thin wire on which the radioactive substance is deposited is used as the source. The α-particles from the source are collimated by a system of slits. The collimated beam is subjected to a homogeneous magnetic field at right angles to the direction of the beam. Under its influence, a slightly divergent beam of α-particles describes semi-circular paths and are focussed at one point on the plate for a given energy. Actually the slit has the appearance of a line of short length, the α-particles are also focussed along a short line on the plate parallel to the slit. α-particles of different energies produce different such lines which have the appearance of the lines in the optical spectra.

Let $B$ be the magnetic induction field, $v$ the velocity and $R$ the radius of the semicircular path of the α-particles. Then we can write

$$Bqv = \frac{Mv^2}{R}$$

or,

$$v = \frac{BqR}{M} \quad \ldots \quad (4.4-1)$$

Rosenblum used magnetic induction fields upto 3.6 T (36,000 gauss). Later Rutherford and others used a similar apparatus for measuring the velocity of the α-particles, but used an ionization chamber instead of a photographic plate as the α-detector. In later years, more improved apparatus were used to measure the velocity of the α-particles accurately. These measurement have yielded the following results.

The velocities of the α-particles emitted by radioactive substances are of the order of 10⁷ m/s. For some radio-elements, only one line is obtained in the α-spectrum on the photographic plate, showing that they emit α-particles of a single velocity. In some cases a number of parallel lines separated from one another (discrete spectrum) are obtained, which shows that a number of different mono-energetic groups of α-particles are emitted by these substances. Each group has a definite velocity; the velocities of the different groups being different.

The kinetic energies of the α-particles emitted from naturally radioactive substances are usually in the range of 4 to 10 MeV.

4.5 Alpha disintegration energy

At the time of α-emission, the emitting nucleus is normally at rest, having zero momentum. Hence the total momentum of the final products in α-disintegration must also be zero. Since the α-particles are emitted with a definite kinetic energy, they must have a definite momentum each. Hence the residual nucleus left after the α-disintegration must have a recoil momentum, equal and opposite to the α-momentum. Thus the residual nucleus must have a certain amount of kinetic energy. Since these nuclei are usually much heavier than the α-particles, their kinetic energy must be small compared to that of the latter (\( \therefore E_R = p^2/2M_1 \)).

The α-disintegration of a nucleus $X$ of mass number $A$ and atomic number $Z$ can be symbolically written as

$$\frac{A}{2}X \rightarrow \frac{A}{2}Y + \frac{Z}{2}He$$

Here $Y$ is the residual nucleus of mass number $A-4$ and atomic number $Z-2$.

If the masses of the α-particle (${}^4He$) and the residual nuclei be $M_\alpha$ and $M_1$, respectively, and $v$ and $v_1$ their respective velocities then conservation of momentum requires that

$$M_\alpha v_\alpha = M_1 v_1$$

If $Q$ is the α-disintegration energy, which is the total energy released in the disintegration process, we can write

$$Q = \frac{1}{2} M_\alpha v_\alpha^2 + \frac{1}{2} M_1 v_1^2$$

$$= \frac{1}{2} M_\alpha v_\alpha^2 + \frac{1}{2} \frac{M_\alpha^2 v_\alpha^2}{M_1}$$

$$= \frac{1}{2} M_\alpha v_\alpha^2 \left( 1 + \frac{M_\alpha}{M_1} \right)$$

The kinetic energy of the α-particle is $E_\alpha = \frac{1}{2} M_\alpha v_\alpha^2$. We then get
The range of the α-particles depends on their initial velocity or kinetic energy. Accurate measurement of the ranges of α-particles of different velocities gives the relationship between the two quantities: $R = R(v)$.

There are various methods for the measurement of the range of the α-particles, some of which are discussed below.

(1) W.H. Bragg, in England, was the first to determine the α-range by measuring the ionization produced by them at different distances from the sources along their path within the medium. His experimental arrangement is shown in Fig. 4.9.

The α-particles emitted by the source S are collimated by a slit within the plate P. A and B are two parallel wire-gauzes with a very small gap between them. The positions of these wire-gauzes can be changed without altering the distance between them.

The collimated beam of α-particles enters the region between A and B. As mentioned earlier (§ 3.1), α-rays can ionize the gas through which they pass. During travel through a gas, an α-particle suffers repeated collisions with the gas atoms. At each collision, a small fraction of its energy is transferred to these atoms which get ionized. Thus a very large number of ion-pairs is produced along the path of the α-particle. Of these, the ion-pairs produced between A and B are attracted towards these due to the potential difference applied between them which gives rise to an ionization current between them. This ionization current can be measured with the help of an electrometer. The potential difference between A and B is so adjusted that all the ion-pairs produced between them are drawn to them, thereby producing a saturation current. This saturation current is obviously proportional to the number of ion-pairs produced between A and B.
Bragg moved the wire-gauzes to different distances from the source and measured the saturation ion-current between them and plotted it as a function of the mean distance of the gauzes from the source. Such a graph is shown in Fig. 4.10.

From Fig. 4.10 it can be seen that the ion-current, which is a measure of the ionizing power of the \( \alpha \)-particles, rises with increasing distance of its travel from the source. The increase is slow at first, but more rapid afterwards. After reaching a maximum the ion current very rapidly begins to go down and falls to zero at a definite distance from the source. This distance is known as the range of the \( \alpha \)-particles.

Fig. 4.10 shows that the steeply falling portion of the ionization current graph (BCD) bends slightly to the right (portion CD) just before the current becomes zero. The bent portion arises due to the straggling of the range. Its origin will be explained later (§ 4.15). The straggling is about 1% to 2% of the range. If we draw a tangent at the point of inflection of the steeply falling portion BCD of the graph, then this intersects the abscissa which gives the so-called ionization extrapolated range \( R \).

The number \( \Delta n/\Delta x \) of ion-pairs produced per unit path of its travel by an \( \alpha \)-particle in a gas at one atmospheric pressure is known as the specific ionization. Here \( \Delta n \) is the number of ion-pairs produced in a distance \( \Delta x \). The specific ionization obviously depends on the distance travelled by the \( \alpha \)-particle from the source. For the Ra\( C \) \( \alpha \)-particles \( (E = 7.68 \text{ MeV}) \), the maximum value of the specific ionization is about 6000 ion pairs per millimetre.

Fig. 4.10 shows that the specific ionization increases as the distance travelled by the \( \alpha \)-particle from the source increases. The reason for this can easily be understood. As the \( \alpha \)-particle moves farther away from the source, its velocity decreases due to loss of energy by ionization of the atoms of the gas. The slower \( \alpha \)-particles spend longer time near the atoms in the gas. So there is a greater probability of their interaction with the electrons in the atom which is the cause of ionization of the atoms (see later). For this reason, when the \( \alpha \)-particles are near the end of their paths the specific ionization is the maximum.

\( \text{(ii)} \) Another method of determining the range of the \( \alpha \)-particles is to take the photographs of their tracks in the Wilson cloud chamber described in Ch. VII. Such a photograph is shown in Fig. 4.11 which shows the tracks of the individual \( \alpha \)-particles emitted by the source. It is obvious from the figure that the straight line tracks of the \( \alpha \)-particles are all of almost the same length. The slight differences between the individual track-lengths are due to the straggling of range mentioned earlier. The mean of the track-lengths gives the mean range \( R \) of the \( \alpha \)-particles.

Fig. 4.11 shows two groups of tracks having two definite lengths. This is due to the fact that the source in this case was a mixture of Th\( C \) and Th\( C' \) which emit \( \alpha \)-particles of two different energies. The figure clearly confirms the statement that the range of the \( \alpha \)-particles is a function of their energy or velocity.

The thickness of the \( \alpha \)-particle tracks shown in the cloud chamber photograph is a measure of the specific ionization mentioned above. Since this is highest near the end of the \( \alpha \)-particle path, the tracks are the thickest near the end. The thinning down of the tracks at the end is due to capture and loss of electrons by the \( \alpha \)-particles, discussed in § 4.13.

\( \text{(iii)} \) H. Geiger and J.M. Nuttall, two of Rutherford's \textit{colleagues}, used a different method for measuring the \( \alpha \)-particle range.

Their experimental arrangement is shown in Fig. 4.12. B is a spherical glass bulb, at the centre of which is placed an \( \alpha \)-source \( S \) mounted at the end of a metal rod. The inner surface of the bulb is lightly coated with silver so that a potential difference can be applied between this surface and the source. The gas pressure inside \( B \) can be controlled as desired.

The \( \alpha \)-particles emitted from \( S \) produce ion-pairs by collision with the gas atoms as they travel towards the inner surface \( C \) of the bulb. The positive and negative ions are attracted by the opposite electrodes, so that an ion current is produced between \( C \) and \( S \), which can be measured by an electrometer \( E \).

The variation of the ion-current with the gas pressure in \( B \) is shown in Fig. 4.13. This shows that at higher pressure, the ion current is a constant. When the pressure is reduced below a
critical value $p_c$, the ion-current begins to decrease. At this pressure, the range of the $\alpha$-particles becomes equal to the radius of the bulb. The reason for this can be understood easily.

At higher gas pressure, when the range of the $\alpha$-particles is less than the bulb radius, they produce the maximum number of ion-pairs, which is possible for them, within the gas inside the bulb. Even if the gas pressure is reduced, there is no change in this number, so that the ion-current remains constant. If, however, the gas pressure is reduced so that the range of the $\alpha$-particles becomes greater than the bulb radius, then the $\alpha$-particles spend only a fraction of their energy in the gas inside the bulb and hit the inner surface of the bulb with the balance of energy which they still possess. Obviously the number of ion-pairs produced by them in this case inside the bulb is less than the maximum possible. So the ion-current decreases with decreasing pressure in this case. At the critical pressure $p_c$ at which the ion-current begins to decrease, the range of the $\alpha$-particles becomes equal to the radius $a$ of the bulb. Since the range is inversely proportional to the gas pressure ($R = 1/\rho$), it is possible to determine the range at S.T.P. when the pressure is $p_i$:

$$R = \frac{p_c}{a}$$

or,

$$R = \frac{ap_c}{p_i}$$

The standard range of the $\alpha$-particles is taken to be equal to the range in pure dry air at 76 cm pressure and 15° C.

(iv) Another method for the measurement of the range of the $\alpha$-particles is to determine the intensity of a collimated beam of $\alpha$-particles at different distances from the source. This remains unchanged with distance till the end of the path of the $\alpha$-particles. M.S. Holloway and M.G. Livingston used this method to determine the a-ranges quite accurately with the help of a shallow ionization chamber as the $\alpha$-detector (1938). More recently, scintillation detectors have been used for this purpose. A typical experimental arrangement is shown in fig. 4.14a. The collimated beam of $\alpha$-particles from a given source falls on the scintillation phosphor (e.g., ZnS) attached to a photomultiplier tube which records a current proportional to the intensity of the $\alpha$-beam.

The distance between the source and the scintillation detector is gradually increased and a graph of the intensity versus the distance obtained. The graph has the appearance shown in Fig. 4.14b. Such a graph is known as the integral range curve. For a monoenergetic beam of $\alpha$-particles, the intensity falls abruptly to zero at a definite distance from the source, which is the range of the $\alpha$-particles. In practice, as shown in Fig. 4.14b, the fall in the intensity $I$ follows the straight line ABC which, though very steep has a finite slope. If all the $\alpha$-particles having the same initial energy made an equal number of identical collisions in the absorber, the intensity would drop to zero abruptly at a distance $d$ equal to the range of the $\alpha$-particles. The portion ABC of the graph would then fall off vertically down. The finite slope shows that there is statistical fluctuation in the number of collisions suffered by the different $\alpha$-particles. The tail of the bent portion (CD) is due to the straggling of the range mentioned above (see § 4.14).

![Diagram](a)

**Fig. 4.14.** (a) Experimental arrangement for obtaining integral range curve.

(b) Variation of the $\alpha$-intensity with distance of the detector from the $\alpha$-source (integral range curve).

When the linear portion ABC of the graph is extrapolated to $z = 0$ intensity, we get the extrapolated range ($R_{ex}$) as shown in the figure. On the other hand, the distance from the source at which the intensity is half the initial intensity is known as the mean range $R$ which is a few millimetres shorter than $R_{ex}$. The mean range has a value such that 50% of the $\alpha$-particles in the incident beam have ranges greater than $R$ while 50% have ranges less than $R$.

For a given energy of the incident $\alpha$-particles, the different ranges defined above have slightly different values. As an example, for the $210^\text{Po}$ $\alpha$-rays ($E_\alpha = 5.3007$ MeV), the following values were obtained:

- Ionization extrapolated range: $R_i = 3.870$ cm
- Extrapolated range: $R_{ex} = 3.870$ cm; Mean range: $R = 3.842$ cm

### 4.7 Range-energy relationship for $\alpha$-particles

From the measured values of the ranges and energies of the $\alpha$-particles, the following mathematical relationship between these two quantities has been established:

$$R = aE^{3/2}$$

...(4.7-1)
This empirical relationship, valid in a limited energy range, is known as Geiger's law.

For $E$ in MeV and $R$ in centimetre, the constant $a = 0.815$

If $v$ be the $\alpha$-velocity in cm/s then since $v \ll \sqrt{E}$, we can write

$$ R = b v^3$$  \hspace{1cm} (4.7-2)

where $b$ is another constant: $b = 9.416 \times 10^{-28}$.

In the case of a solid, the range $R_s$ (in centimetre) is related to the range $R$ in air as follows:

$$ R_s = \frac{0.312 R A^{1/2}}{\rho}$$

where $\rho$ is the density of the solid of mass number $A$.

In the following table, the mean ranges in air of the $\alpha$-particles from some radioactive elements and their kinetic energies are listed. In the last column, the ratios $R/E^{1/2}$ are given, which are found to be approximately constant for different substances, showing the approximate validity of Geiger's law. Actually the law holds fairly well between the energies 3 to 7 MeV. For higher and lower energies, more appropriate relationship are $R \propto v^4$ and $R \propto v^{3/2}$ respectively.

<table>
<thead>
<tr>
<th>Radioactive Isotope</th>
<th>$\alpha$-energy $E$ (MeV)</th>
<th>Mean range $R$ (centimetres)</th>
<th>$R/E^{1/2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{210}$Po</td>
<td>5.3007</td>
<td>3.842</td>
<td>0.315</td>
</tr>
<tr>
<td>$^{222}$Rn</td>
<td>5.4561</td>
<td>4.051</td>
<td>0.315</td>
</tr>
<tr>
<td>$^{218}$Po(RaA)</td>
<td>5.9982</td>
<td>4.657</td>
<td>0.317</td>
</tr>
<tr>
<td>$^{220}$Rn(ThEm)</td>
<td>6.2823</td>
<td>5.004</td>
<td>0.318</td>
</tr>
<tr>
<td>$^{219}$Rn(AcEm)</td>
<td>6.542</td>
<td>5.240</td>
<td>0.313</td>
</tr>
<tr>
<td>..</td>
<td>6.807</td>
<td>5.292</td>
<td>0.320</td>
</tr>
<tr>
<td>$^{215}$Po(AcA)</td>
<td>7.383</td>
<td>6.457</td>
<td>0.322</td>
</tr>
<tr>
<td>$^{214}$Po(RaC')</td>
<td>7.6804</td>
<td>6.907</td>
<td>0.324</td>
</tr>
<tr>
<td>..</td>
<td>8.2271</td>
<td>7.793</td>
<td>0.327</td>
</tr>
<tr>
<td>$^{212}$Po(ThC')</td>
<td>8.7801</td>
<td>8.570</td>
<td>0.329</td>
</tr>
<tr>
<td>$^{214}$Po(RaC')</td>
<td>9.0649</td>
<td>9.04</td>
<td>0.331</td>
</tr>
<tr>
<td>$^{212}$Po(ThC')</td>
<td>9.4923</td>
<td>9.724</td>
<td>3.333</td>
</tr>
<tr>
<td>..</td>
<td>10.5432</td>
<td>11.58</td>
<td>0.338</td>
</tr>
</tbody>
</table>

**Table 4.1**

### 4.8 Geiger-Nuttall law

Geiger and Nuttall (1911) discovered an empirical relationship between the ranges of the $\alpha$-particles and the disintegration constants of the naturally $\alpha$-active substances emitting them. This is known as Geiger-Nuttall law which can be expressed as

$$ \log \lambda = A + B \log R$$  \hspace{1cm} (4.8-1)

$A$ and $B$ are constants. According to this law, the $\alpha$-particles emitted by substances having larger disintegration constants (i.e., shorter half-lives) have longer ranges and vice-versa.

Eq. (4.8-1) shows that the graph of $\log \lambda \lambda$ and $\log R$ is a straight line with a slope $B$. For different radioactive series, different straight lines are obtained, which are parallel to one another, so that $B$ is the same for all of them (see Fig. 4.15).

Using Eq. (4.8-1), it is possible to determine the disintegration constant and hence the half-life $\tau$ of a radioactive substance if the range of the $\alpha$-particles emitted by it is known accurately (see § 3.12-iv).

Since the range $R \propto E^{3/2}$, the Geiger-Nuttall law can also be written as

$$ \log \lambda = C + D \log E$$

where $C$ and $D$ are two other constants.

Since the half-life $\tau = \ln 2/\lambda$, Geiger-Nuttall law can also be expressed by relating the variation of $\log \tau$ with $\log R$ or $\log E$. In this case also we shall get straight line graphs, but with negative slopes.

It may be noted that the ranges of the $\alpha$-particles from different radioactive isotopes are of the order of few centimetres in air. Their
half-lives, on the other hand, extend from less than one-millionth of a second to more than billion ($10^9$) years. As an example, the half-life of ThC' is $3 \times 10^{-7}$ s, while that of $^{232}$Th is $1.39 \times 10^{10}$ y. The ranges of the $\alpha$-particles emitted by them are 8.57 cm and 2.49 cm respectively. The corresponding energies are 8.78 MeV and 3.97 MeV respectively. This shows that for an increase of the $\alpha$-energy by a factor of 2.24, the half-life decreases by a factor of $10^{24}$. Such an enormous change of the half-life due to relatively small change of the $\alpha$-energy can be explained by the quantum mechanical theory of potential barrier penetration (see §4.9).

It may be noted that Geiger-Nuttall law does not give a very accurate representation of the variation of the half-life with $\alpha$-energy. Fig. 4.16 shows a plot of $(\log \tau)$ against $\alpha$-energy for even Z-even $N$ isotopes ($N$ = neutron number) for different elements (different Z). It shows a linear dependence of $(\log \tau)$ on $E_\alpha^{-1/2}$.

Similar plot in the case of odd Z-odd N, odd Z-even N or even Z odd $N$ nuclei show a retardation by a factor of about 100, compared to the even-even nuclei in (cf. emission probability (see later).

4.9 Theory of $\alpha$-disintegration

Experiments on the scattering of the $\alpha$-particles discussed in Ch. I show that an $\alpha$-particle, while approaching the nucleus, is acted upon by the Coulomb repulsive potential $V_c \propto 1/r$ up to almost the nuclear surface. The anomalous scattering of the $\alpha$-particles observed at larger angles in the case of some lighter atoms shows that the nature of the force is different from the Coulomb repulsion for very close distance of approach (see Ch. II). Our present day knowledge regarding the nuclear structure tells us that the neutrons and protons forming the nucleus of an atom are very strongly attracted to one another, giving rise to a strongly bound structure, known as the nucleus of the atom. This attractive force acts up to distances of the order of $2 \times 10^{-15}$ m. Thus the specifically nuclear force holding the neutrons and protons together is a strong short range force. The $\alpha$-disintegration of the heavy nuclei shows that within such nuclei, two protons and two neutrons sometimes form a cluster known as the $\alpha$-particle. As long as an $\alpha$-particle is within the nucleus, it must be acted upon by this strong short range specifically nuclear attractive force. However once it is outside the nucleus, the Coulomb repulsion due to the residual nucleus of positive charge $(Z - 2)$ acts on it, where $+Ze$ is the charge of the original nucleus. Thus the potential energy of an $\alpha$-particle at various distances from the centre of the atom has the appearance shown in Fig. 4.17.

Outside the nucleus, the Coulomb potential energy $V_c = \frac{2(Z - 2)e^2}{4\pi\varepsilon_0 r}$ is positive and is represented by the line AEB, where $\varepsilon_0 = \frac{1}{36\pi\varepsilon}$ is the permittivity of vacuum. For $r < R$, the nuclear radius, this suddenly changes into an attractive potential, represented by the line BCD. The exact nature of this potential is not known. All we know that it is a strong short range attractive potential, so that it must have a very steep positive slope, since the force is the negative gradient of the potential energy: $F = -\nabla V$. Thus for a positive slope, the force is attractive (negative).

Also the steepness of the potential curve shows that it must be very strong.

The transition from the repulsive Coulomb potential to the attractive potential in the nucleus takes place at the nuclear surface $r = R$ where the Coulomb potential has the largest value (the point B in Fig. 4.17).

\[ V_s = \frac{2(Z - 2)e^2}{4\pi\varepsilon_0 R} \]

For the case of $^{222}$Ra (Z = 88), the value of $V_s$ comes to be 34 MeV. The $\alpha$-disintegration energy $Q_\alpha = 4.88$ MeV in this case is much lower than $V_s$. This is the energy the $\alpha$-particle has in the disintegrating nucleus. Classical mechanics requires that for an $\alpha$-particle to escape from the nucleus or to enter it from outside its energy must at least be equal to $V_s$. If it is lower, then in some region between the two curves AB and BC, the potential energy of the $\alpha$-particle will be greater than its total energy. This is a classically forbidden region, since the kinetic energy of the $\alpha$-particle would be negative in this region. As an example, if the $\alpha$-disintegration energy $Q_\alpha$ is represented by the line FE, then obviously $Q_\alpha < V_s$. As long as the $\alpha$-particle is inside the nucleus (i.e., in the region OR), or at points to the right of EH outside the nucleus, its potential energy is less than its total energy, so that its kinetic energy is positive. However in the region RH, known as the potential barrier region, the total energy of the $\alpha$-particle is less than its potential energy. Its kinetic energy is negative.
here, so that this is the classically forbidden region. An α-particle in this case, can neither escape from nor enter into the nucleus.

However, quantum mechanically, such classically forbidden phenomenon may occur. In quantum mechanics, the particle is represented by a wave, obeying the Schrödinger wave equation. We can write down the wave equation for the different regions by substituting the corresponding potentials acting on the α-particle in these regions. If these equations are solved with appropriate boundary conditions, then it is found that an α-particle initially inside the nucleus has a finite probability of coming out of it. George Gamow, R.W. Gurney and E.U. Condon were the first to provide a satisfactory explanation of the α-decay in this manner (1928) with the help of quantum mechanics. We can picture the escape of the α-particle from the nucleus as if they leak out through tunnels in the potential barrier. Hence the effect is called quantum mechanical tunnel effect. The theory gives a mathematical relationship between the initial α-energy and the half-life of the disintegrating nucleus, which can be evaluated by determining the amplitudes A and C by matching the solutions (4.9-3) at the boundaries x = 0 and x = a. When this is done we get the transmission coefficient (see Eq. 11.4-14 in Vol. I)

$$T = \frac{\exp(-\beta a)}{\exp(-2\beta a)}$$  \hspace{1cm} (4.9-4)

under the approximation E ≪ V₀.

Eqs. (4.9-3) and (4.9-4) show that the transmission coefficient is very strongly influenced by the barrier thickness a and the energy E of the particle for a given barrier height V₀, since both a and E occur in the exponent of the exponential. Small changes in either of them or both change the value of T by a very large factor which, shows qualitatively the reason for very large variation of the α-disintegration probability due to small variation of the α-energy (see § 4.8).

Let us take a numerical example. Consider the case of V₀ = 15 MeV, a = 2 × 10⁻¹⁴ m and E = 5 MeV. Then for M = Mₐ = 4 × 1.66 × 10⁻²⁷ kg,

$$\beta^2 = \frac{2M}{\hbar^2} (V₀ - E)$$

$$= 2 \times 4 \times 1.66 \times 10^{-27}$$

$$\left(\frac{1.05 \times 10^{-34}}{2}\right)^2 \times 10 \times 1.6 \times 10^{-13}$$

$$= 1.927 \times 10^{30}$$

$$\beta = 1.388 \times 10^{15}$$

and 

$$2\beta a = 55.53$$

and 

$$\exp(-\beta a) = 7.649 \times 10^{-25}$$

$$T = 16 \times 5 \times \frac{10}{15 \times 15} \times 7.649 \times 10^{-25}$$

$$= 2.72 \times 10^{-24}$$

The above calculations show that the particle has the probability of about 2.7 in 10⁻²⁴ collisions against the barrier wall to penetrate through the barrier.

In order to apply the above considerations to the problem of α-disintegration of a nucleus, we note that an α-particle of velocity vₐ in the nucleus...
Inside the nucleus of radius \( R = 10^{-14} \) m takes a time \( t = 2R/v_{in} \) to cross the nucleus from one end to the other (see Fig. 4.19)

\[
v_{in} = \frac{1.552 \times 10^{10}}{2 \times 10^{-14}} = 7.76 \times 10^{20} \text{ s}^{-1}
\]

So the probability of escape of the \( \alpha \)-particle per second which, is equal to the decay constant \( \lambda \) is \( P = \lambda = nT \). Putting the numerical values of \( n \) and \( T \) calculated above, we then get

\[
\lambda = 7.76 \times 10^{20} \times 2.72 \times 10^{-24} = 2.11 \times 10^{-3}
\]

So the mean life time against \( \alpha \)-decay is

\[
\tau_m = \frac{1}{\lambda} = \frac{1}{2.11} = 474 \text{ s} = 7 \text{ min } 54 \text{ s}
\]

If the barrier width \( a \) is increased by 20% (\( a = 2.4 \times 10^{-14} \) m), then \( 2 \beta a = 66.63 \). This gives the mean life as

\[
\tau_m = 3.13 \times 10^{7} \text{ s} = 0.993 \text{ h}
\]

This is about \( 6.6 \times 10^{9} \) times larger than in the previous case. On the other hand, if the energy of the particle is 20% less, then we get \( \tau_m = 9.018 \text{ s} = 2 \text{ h } 30.3 \text{ min} \) which is about 19 times as large.

These calculations bear out the fact that slight changes in either the width of the barrier or of the energy of the particle, make large changes in the value of the mean life \( \tau_m \) and hence of the decay constant \( \lambda \). In the case of the more realistic barrier shape shown in Fig. 4.17, change in the energy of the \( \alpha \)-particle also changes the barrier thickness (RH in the figure). Hence the mean life of \( \alpha \)-disintegration is enormously changed by slight change in the \( \alpha \)-energy in agreement with Geiger–Nuttall law.

### 4.10 More realistic theory of \( \alpha \)-disintegration

The rectangular potential barrier is an idealized barrier. In the actual process of the \( \alpha \)-disintegration of a nucleus, the potential barrier, which the \( \alpha \)-particle has to penetrate, is much more complex and has the approximate shape shown in Fig. 4.17. Before we consider the penetration of the potential barrier in this case, let us first consider the penetration of a barrier of an arbitrary shape shown in Fig. 4.20.

\[
T = \exp(-G) \quad (4.10-1)
\]

where

\[
G = \left[ \frac{\beta}{\hbar} \int_a^b \sqrt{2M[V(x)] - E} \, dx \right] \quad (4.10-2)
\]

where the barrier extends from \( x = a \) to \( x = b \).

Replacing \( x \) by \( r \) we have in present case
\[ V = V(r) = \frac{(Z-2)e^2}{2\pi\varepsilon_0 r} \]

Ze is the charge of the parent nucleus undergoing by the \(\alpha\)-disintegration. We then get.

\[
G = \frac{2}{\hbar} \left[ \sqrt{2M} \left( \frac{(Z-2)e^2}{2\pi\varepsilon_0 r} - E \right) \right] dr
\]

\[
= \frac{2}{\hbar} \left\{ \frac{(M(Z-2)e^2)}{\pi\varepsilon_0} \right\} \int_r^0 \left[ \frac{1}{r} - \frac{2p\varepsilon_0 E}{r(Z-2)e^2} \right] dr \quad \ldots (4.10-3)
\]

The limits of integration are from \(r = R\) to \(r = b\) at which point the Coulomb energy is equal to the \(\alpha\)-energy, so that

\[
\frac{2(Z-2)e^2}{4\pi\varepsilon_0 b} = E
\]

which gives

\[
\frac{1}{b} = \frac{2\pi\varepsilon_0 E}{(Z-2)e^2}
\]

Hence we have

\[
G = \frac{2}{\hbar} \left\{ \frac{M(Z-2)e^2}{\pi\varepsilon_0} \right\}^{1/2} \int_r^0 \left( \frac{1}{r} - \frac{1}{b} \right) dr
\]

\[
= \frac{2}{\hbar} \left\{ \frac{M(Z-2)e^2}{\pi\varepsilon_0} \right\}^{1/2} \left[ \frac{\pi}{2} \right] \frac{1}{b} \left\{ \frac{M(Z-2)e^2}{\pi\varepsilon_0} \right\}^{1/2} \ldots (4.10-4)
\]

Usually \(b >> R\) so that \(R/b \ll 1\). In this case the term \((R/b - R^2/b^2)^{1/2}\) on the r.h.s. of the above expression may be neglected. Also \(\cos^{-1}(R/b)^{1/2} = \pi/2\). So we get

\[
G = \frac{2}{\hbar} \left\{ \frac{M(Z-2)e^2}{\pi\varepsilon_0} \right\}^{1/2} \left[ \frac{\pi}{2} \right] \frac{1}{b} \left\{ \frac{M(Z-2)e^2}{\pi\varepsilon_0} \right\}^{1/2} \ldots (4.10-5)
\]

If \(v\) be the \(\alpha\)-velocity, we have

\[
E = \frac{1}{2} M v^2 = \frac{(Z-2)e^2}{2\pi\varepsilon_0 b} \frac{p}{\varepsilon_0 M v^2}
\]

which gives

\[
b = \frac{(Z-2)e^2}{\pi\varepsilon_0 b M v^2}
\]

and

\[
G = \frac{\pi}{\hbar} \left\{ \frac{M(Z-2)e^2}{\pi\varepsilon_0} \right\}^{1/2} \left( \frac{Z-2)e^2}{\pi\varepsilon_0 M v^2} \right)^{1/2} = \frac{\pi}{\hbar} \left( \frac{(Z-2)e^2}{\pi\varepsilon_0} \right)^{1/2} \ldots (4.10-6)
\]

Thus the barrier penetration probability becomes

\[
T = \exp (-G) = \exp \left\{ \frac{(Z-2)e^2}{\varepsilon_0 \hbar v} \right\}
\]

As an example, for \(E = 5\) MeV, we get

\[
v = \frac{\sqrt{2E}}{M} = 1.55 \times 10^7 \text{ m/s}
\]

Then for \(Z-2 = 86\), we get

\[
G = \frac{86 \times (1.6 \times 10^{-19}) \times 36\pi}{10^{-9} \times 1.05 \times 10^{-34} \times 1.55 \times 10^7} = 153
\]

On the other hand for \(E = 10\) MeV, we get

\[
v = \sqrt{2} \times 1.55 \times 10^7 = 2.19 \times 10^7 \text{ m/s}
\]

and

\[
G' = \frac{153}{12} = 108
\]

So the ratio of \(\alpha\)-emission probabilities in the two cases is

\[
\frac{T_{10}}{T_5} = \frac{\exp (-G')}{\exp (-G)} = \exp (G - G') = \exp (45) = 3.49 \times 10^9
\]

Thus an increase in the \(\alpha\)-energy by the factor of 2 increases the \(\alpha\)-emission probability by a factor of about \(3.5 \times 10^9\). This is in qualitative agreement with Geiger-Nuttall law (§ 4.8).

Though the expression for the barrier penetration probability given above explains qualitatively the observed dependence of the \(\alpha\)-decay half life on the energy of the \(\alpha\)-particles quantitative calculations do not reproduce the observed half-lives in actual cases.

As in the case of the rectangular barrier penetration problem discussed earlier, we can write the \(\alpha\)-disintegration constant as

\[
\alpha = \frac{\nu_{\ln}}{2R} \exp (-G)
\]

Hence

\[
\ln \lambda = \ln \frac{\nu_{\ln}}{2R} - G = \ln \frac{\nu_{\ln}}{2R} - \frac{(Z-2)e^2}{\varepsilon_0 \hbar v}
\]

using Eq. (4.10-6).

Since both \(\nu_{\ln}\) and \(R\) do not vary much for \(\alpha\)-emitters amongst the naturally radioactive series, in \(\nu/2R\) may be taken to be almost a constant which we put equal to \(A\). Hence we get

\[
\ln \lambda = A - B \frac{(Z-2)}{\nu}
\]

where \(B = e^2/\hbar\) \(\varepsilon_0 = \text{constant}\) \ldots (4.10-10)
Eq. (4.10-9) may be regarded as a theoretical form of Geiger-Nuttall rule. A plot of \( \ln \lambda \) against the reciprocal of the \( \alpha \)-velocity should give a straight line for a given \( Z \). This is actually found to be the case for even \( Z \) and even \( A \) isotopes (see Fig. 4.21a). Alternatively, the results can also be shown by plotting \( \log \tau \) against \( \alpha \)-disintegration energy, as in Fig. 4.21b where \( \tau \) is the half-life. Using Eq. (4.10-9) it is easily seen that

\[
\log \lambda = A' + \frac{B'}{\sqrt{E}}
\]

where \( A' \) and \( B' \) are constants and \( E \) is the \( \alpha \)-disintegration energy, assuming \( v_\alpha = v \).

The graphical results shown in Figs. 4.21a and b provide fairly good confirmation of the simple theory discussed above.

4.11 Modification of the simple theory; hindrance and formation factors

The above simple theory needs modification on three counts:

(i) Effect of angular momentum change: It is observed that though the theory discussed above gives some agreement with the observed results for even-even nuclei, there are great discrepancies for even-odd, odd-even and odd-odd nuclei. For example, in the case of an even-odd nucleus, the half-life in many cases is 100 to 1000 times larger than that for an even-even nucleus of the same \( Z \) and same energy. In no case is the half-life shorter than for the corresponding even-even value. The ratio of the actual half-life to that predicted for the even-even nuclides is known as the hindrance factor:

\[
\text{Hindrance factor} = \frac{\tau_{\text{obs}}}{\tau_{\text{ee}}} = \frac{\lambda_{\text{ee}}}{\lambda_{\text{obs}}}
\]

The patterns for the odd-even and odd-odd nuclei are similar. The average hindrance factor for the latter is perhaps greater than for the former.

In the case of \( \alpha \)-disintegration of an even-even nucleus in the ground state leading to the formation of the even-even daughter nucleus in the ground state, the emitted \( \alpha \)-particle does not carry any orbital angular momentum \( (l = 0) \), since both the parent and product nuclei have \( l = 0 \) and same parity (even). In those cases in which the \( \alpha \)-particles are emitted with non-zero \( l \), they are subjected to the action of the centrifugal potential \( V_c \) in addition to the Coulomb potential \( V_c \) where

\[
V_c = \frac{\pi^2}{2M} \frac{l(l+1)}{r^2}
\]

\[
V_c = \frac{2(Z-2)e^2}{4 \pi \varepsilon_0 r}
\]

Usually \( V_c \ll V_c \), as can be easily checked by calculating the ratio

\[
\sigma = \frac{V_c}{V_c} = \frac{\sigma_0}{V_c} \quad \text{for naturally radioactive substances for which } Z \text{ lies in the range 30 to 92. Taking } Z = 88 \text{ and } r = 10^{-14} \text{ m we get}
\]

\[
\sigma = \frac{\pi^2 \varepsilon_0 l(l+1)}{Me^2r(Z-2)} = 0.0021 l(l+1)
\]
So for the usual values of \( I < 10 \), \( \sigma < 1 \). Thus the effect of the centrifugal potential is rather small. To calculate the probability of \( \alpha \)-disintegration, we have to take the total potential \( V = V_c + V_t \) in Eq. (4.10-2) for the barrier penetration probability \( T \):

\[
T = \exp \left( -\frac{G}{\hbar} \right) = \exp \left[ -\frac{2}{\hbar} \int \left[ 2M(V_c + V_t - E) \right]^{1/2} dr \right]
\]

So

\[
G = \frac{2}{\hbar} \sqrt{2M} \int \left[ \frac{2(Z-2)\epsilon^2}{4\pi\epsilon_0 r} + \frac{\pi^2}{2M} \frac{l(l+1)}{r^2} - E \right]^{1/2} dr \quad \text{(4.11-2)}
\]

Numerical calculations using the above formula (4.11-2) give the following values of the hindrance factor \( \lambda_0/\lambda = \tau_r/\tau_0 \) in the case of \( Z = 88 \) and \( E = 4.88 \text{ MeV} \) for different values of \( I \) (see Theoretical Nuclear Physics by Blatt and Weisskopf).

<table>
<thead>
<tr>
<th>( l )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_0/\lambda )</td>
<td>1.43</td>
<td>2.7</td>
<td>7.3</td>
<td>27</td>
<td>141</td>
<td>909</td>
<td></td>
</tr>
</tbody>
</table>

The calculations show that the \( \alpha \)-disintegration constants with \( l = 0, 1, 2 \) and 3 are equal to one another within the likely error of the theoretical estimate for \( l = 0 \).

(ii) Effect of non-spherical potential: The most important reason for hindered \( \alpha \)-decay lies in the non-spherical nature of the nuclei undergoing \( \alpha \)-decay. As will be seen in Ch. IX the nucleons in a non-spherical nucleus move in a non-spherical potential, known as the Nilsson potential. In such a potential the quantum number \( K \) representing the component of the angular momentum along the symmetry axis of a spheroidal nucleus happens to be a good quantum number for the ground state and for all the rotational levels. Unhindered decays are those for which \( K \) does not change in this case. For the even-even nuclei, this is always the case.

The situation is different for the odd \( A \) nuclei. The extra unpaired nucleon in such a nucleus may have different values of \( K \) in the ground states of the parent and product nuclei. If, however, one of the low lying excited states of the product nucleus has the same \( K \) for an unpaired nucleon as that of the ground state of the parent nucleus, then the decay to this state will be unhindered, compared to the ground state decay of such an odd \( A \) nucleus.

The \( \alpha \)-decay of \( ^{241}\text{Am} \) (\( Z = 95 \)) is a case to the point. The last odd nucleon in this nucleus occupies the ground state (\( 5/2^- \)). i.e., \( K = 5/2^- \) and parity is odd for this state. In the daughter nucleus \( ^{237}\text{Np} \), the last unpaired nucleon occupies the level (\( 5/2^+ \)) in the ground state. The (\( 5/2^- \)) state is an excited state in this nucleus. Thus the decay from the ground state (\( 5/2^- \)) of \( ^{241}\text{Am} \) to the ground state of the product \( ^{237}\text{Np} \) (\( 5/2^+ \)) is hindered very greatly. The transition mostly takes place to the excited state (\( 5/2^- \)). The ratio of the two transition probabilities is 85 to 39.

(iii) Formation factor: It is certain that the \( \alpha \)-particles are not permanent constituents of the nuclei. In any given nuclear state, the configuration of the nucleons are continually changing. Some configuration will have greater probability than others. If \( \psi_i \) be the wave function for the configuration \( i \) then the wave function of the nuclear state is \( \Sigma \alpha_i \psi_i \); \( p_i = |\alpha_i|^2 \) is the probability for the configuration \( i \). In some of these configurations, two neutrons and two protons will come together to form an \( \alpha \)-particle momentarily and there is small probability that this cluster will be emitted as such. The probability of emission of an \( \alpha \)-particle of energy \( E_i \) from the configuration \( i \) can thus be divided into two parts: \( \lambda_0 = p_i \lambda_i^0 \) where \( \lambda_i^0 \) is the probability of the emission of the \( \alpha \)-particle of energy \( E_i \) already formed. \( p_i \) is known as the formation factor. \( \lambda_i^0 \) is the same as \( \lambda \) for even-even nuclei. We have seen how this can be estimated. Agreement with experimental results is obtained if we assume \( p_1 = 1 \), \( R = r_0 A^{1/3} \) with \( r_0 = 1.57 \text{ fm} \) and a square well nuclear potential of depth \( V_0 \) such that \( E + V_0 = 0.52 \text{ MeV} \). The \( \alpha \)-particle in these calculations is assumed to be a point particle. The above value of \( r_0 \) is found to be much higher than estimated from electron scattering experiments which give \( r_0 = 1.2 \text{ fm} \). Agreement with the measured values of \( \lambda \) can then be obtained by assuming the \( \alpha \)-particle not to be a point particle. Writing \( R = R_a + r_0 A^{1/3} \) with \( r_0 = 1.2 \text{ fm} \), it is found that the \( \alpha \)-particle radius \( R = 2.2 \text{ fm} \) gives good agreement between the theoretical and experimental values of \( \lambda \) with \( p_i = 1 \). To decrease \( p_i \) the value of \( R_a \) to be assumed would be unreasonably high. The great regularity in the empirical values of \( R \) and \( E + V_0 \) shows that whatever the value of \( p_i \) may be, it is fairly constant for most even-even nuclei.

In an alternative formulation G. H. Winslow (1954) suggested a repulsive core potential inside the nucleus which would exclude the \( \alpha \)-particle from the interior of the nucleus. According to this surface-well potential model, the \( \alpha \)-particle ceases to exist as such if more than half of \( r \) is within the nucleus. The formation factor \( p_i \) in this model is to be interpreted as the probability at any time that an \( \alpha \)-particle will be found with its centre in the region \( R_0 < r < (R_0 + R_a) \) where \( R_0 \) is the radius of the repulsive core and \( R = (R_0 + R_a) \) is the outer radius of the surface-well potential. The formation factor calculated according to this model comes out to be very low \((- 10^{-2} \text{ to } 10^{-3}) \).

The formation factor plays a leading role in causing the longer life times which seem to be characteristic of nuclides with odd protons or neutrons. The hindrance factor is not directly related to the formation factors. Since the formation factor of an even-even nuclide is not unity, the hindrance factor may be taken to be a number which is of the same order as the ratio of the formation factor of an even-even nucleus to that of the actual nucleus considered.

4.12 Fine structure of the \( \alpha \)-ray spectra; Long range \( \alpha \)-particles

We have seen that the \( \alpha \)-particles emitted by a radioactive substance are usually mono-energetic. This shows that they are emitted from one
definite energy state of the parent nucleus to a definite energy state of the product.

In the case of some nuclei, more than one group of α-particles, each of a definite energy, are found to be emitted by the same radioactive substance. This can happen due to the transitions from the different discrete states (excited or ground) of the parent nucleus to the different discrete states of the product nucleus. We can group such transitions into two classes:

(a) In this case the transitions take place from the different excited states of the parent nucleus \(^{A}X\) to the ground state of the product nucleus \(^{A+4}Y\) as shown in Fig. 4.22. The energies of these α-particles, \((E'_{\alpha}, E''_{\alpha}, \ldots)\) are obviously greater than the energy \(E_{\alpha}\) of the main group of α-particles produced by the transition from the ground state of the parent nucleus to the ground state of the product. These are called the long-range α-particles. Their intensities are small compared to the intensity of the main group. They are observed in the α-disintegration of the isotopes Ra\(^{226}\), ThC\(^{228}\), etc., which have very short half-lives. For example, ThC\(^{228}\) (\(t = 3 \times 10^{-7}\) s) emits the α-groups shown in Table 4.2.

\[ \begin{array}{|c|c|}
\hline
\alpha\text{-energy (MeV)} & \text{Relative Intensity} \\
\hline
8.947 & 10^8 \text{ (main group)} \\
9.673 & 34 \\
10.744 & 190 \\
\hline
\end{array} \]

The table shows that the intensities of the second and third group of α-particles are small compared to that of the main group of energy 8.947 MeV. The energies of the former are considerably higher than that of the main group.

The reasons for the facts stated above can be understood as follows. The excited parent nucleus usually goes down to its own ground state by the emission of γ-rays of different energies, either directly or by cascade, as shown in Fig. 4.22, and then undergoes α-disintegration. The probability of such γ-transitions is much higher than that of α-emission from this state directly to the ground state of the product nucleus. The mean life of a nuclear excited state against γ-transition is \(\tau_{\gamma} = 10^{-13}\) s, while the α-disintegration life-times of these nuclei are of the order of \(10^{-4}\) to \(10^{-2}\) s. Since the α-emission half-life of ThC\(^{228}\) is \(t = 10^{-7}\) s, the ratio of the probabilities of α to γ emission will be

\[ \lambda_{\alpha} : \lambda_{\gamma} = \tau_{\gamma} : \tau_{\alpha} = 10^{-13} : 10^{-7} = 1 : 10^6 \]

This is in qualitative agreement with the observed ratio of the intensities of the main α-group and the others.

Obviously this type of α-emission from the excited nuclear states can occur only if the α-decay half-lives are very short as in the C\(^{14}\) products in the naturally radioactive series. Because they have very short lives, the corresponding α-energies must be fairly high according to Geiger-Nuttall law. Hence in this case the α-particles have long ranges.

(b) In some cases α-emission takes place from the ground state of the parent nucleus \(^{A}X\) to the different states of the product nucleus \(^{A+4}Y\) as shown in Fig. 4.23. If the product nucleus is formed in an excited state in such a transformation it subsequently emits γ-rays to make transition to its ground state.

The probabilities of the emission of the different α-ray groups are comparable in this case, so that these groups have comparable intensities, as shown in Table 4.3. These α-rays give rise to the so-called fine-structure of the α-ray spectra. The main α-group in this case arises due to transition between the ground states of the parent, and the product nuclei and have the highest energy. As an example, α-ray groups from ThC\(^{228}\) (\(t = 60.5\) min) have the energies and intensities shown in Table 4.3.

\[ \begin{array}{|c|c|}
\hline
\alpha\text{-energy (MeV)} & \text{Relative Intensity} \\
\hline
6.200 & 27.2 \text{ (main group)} \\
6.160 & 69.8 \\
5.872 & 1.80 \\
5.782 & 0.16 \\
5.708 & 1.10 \\
\hline
\end{array} \]

In both the cases discussed above, definite correlations are observed between the energies of α-rays and the associated γ-rays. We shall discuss about this later (see § 6.8).

**Energy-loss of heavy charged particles in matter**

We have seen that the α-particles from naturally radioactive substances produce intense ionization in matter. Due to this they can travel only a very short distance in matter, rapidly losing energy by ionization. This is the characteristic of all energetic heavy charged particles, e.g., protons, deuterons, \(^{3}H\) and \(^{3}He\) ions in addition to the α-particles. So in this section we shall investigate the theory of energy-loss by heavy charged particles in matter. The theory was first worked out by Niels Bohr, which was later improved upon by Hans Bethe, Felix Bloch and others by the application of quantum mechanics.
When an energetic heavy charged particle travels through matter, there is momentary electrostatic interaction between it and the electrons in the atoms by which it is passing. As a result, each electron gains some momentum and kinetic energy. By calculating the total amount of energy gained by all the electrons surrounding the path of travel of the ions through the medium, it is possible to calculate the rate of energy-loss by the ion.

Suppose a heavy ion of charge $z_e$, mass $M$ and velocity $v$ travels along the straight line $X'X$ shown in Fig. 4.24. Let $m$ and $-e$ be the rest mass and charge of the electron. We take $M \gg m$.

Suppose the ion travels by an electron $Q$ bound in an atom. Let $QO$ be the foot of the perpendicular drawn on the path of travel of the ion from $Q$. Then $QO = b$ is the impact parameter. Let $O$ be taken as the origin of the coordinate system and $R$ the position of the ion at some instant $t$. Then the distance of the ion from the electron at the instant $t$ is

$$QR = r = \sqrt{x^2 + b^2}$$

We make the following assumptions:

(a) The velocity of the electron in the atomic orbit $v_e \ll v$, the velocity of the ion. If $E_e$ and $E_i$ be the energies of the electron and the ion, the above condition gives us

$$\frac{E_i}{M} \gg \frac{E_e}{m} \quad \text{or} \quad \frac{E_i}{m} \gg \frac{E_e}{m}$$

For example if the medium is nitrogen gas, then the energy of its K electron is $\sim 400$ eV. So for a proton ($M/m = 1836$), the above condition requires that the minimum energy of the ion must be greater than about $1$ MeV.

(b) The collision between the ion and the electron may be treated classically. If $p_i$ be the momentum of the ion, then the above condition requires that $p_i \times b \gg \hbar$ where $\hbar = h/2\pi$, $h$ being Planck's constant. This means that the momentum of the ion and the impact parameter must both be large. If the above condition does not hold, so that $p_i \times b \sim \hbar$, then the effect of the uncertainty principle will come into play, so that the collision has to be treated quantum mechanically.

\[ F = \frac{ze^2}{4\pi \varepsilon_0 r^2} \cos \theta \]

The $x$-component of this force is

$$F_x = F \cos \varphi = -F \cos \theta$$

where $\theta$ is the angle made by $QR$ with the $x$-axis and $\varphi = \pi - \theta$.

If we consider two points on the line $X'OX$ equidistant from $O$ on either side, then the $x$-components of the force acting on the electron at these two positions of the ion being equal and opposite, will cancel each other. The underlying assumption here is that because of the very high velocity of ion, the forces on the electron due to the ion at these two positions act at almost the same instant of time. Thus if we take different such pairs of points equidistant from O on the $x$-axis, then the $x$-components of the force on the electron due to the ion for each such pair cancel out, so that the net $x$-component of the force on the electron becomes zero. Hence we have to consider only the component of the force perpendicular to the $x$-axis ($F_y$).

From Fig. 4.24, we get

$$F_y = F \sin \varphi = F \sin \theta = \frac{ze^2}{4\pi \varepsilon_0 r^2} \sin \theta$$

The momentum gained by the electron due to this force acting for a time $dt = dx/v$ is $F_y dt = \frac{ze^2}{4\pi \varepsilon_0 r^2} \sin \theta \frac{dx}{v}$.

Since $x = b/\tan \varphi = -b/\tan \theta$, $dx = b d\theta/\sin^2 \theta$. Also since $r = b/\sin \theta$, we get

$$F_y dt = \frac{ze^2 \sin \theta d\theta}{4\pi \varepsilon_0 b v}$$

Eq. (4.13-1) when integrated from $\theta = 0$ to $\theta = \pi$ gives the momentum gained by the electron perpendicular to the path of the ion:

$$q = \int_0^\pi \frac{ze^2}{4\pi \varepsilon_0 b v} \sin \theta d\theta = \frac{ze^2}{2\pi \varepsilon_0 b v}$$

Consider two coaxial cylinders of radii $b$ and $(b + db)$ and length $Ax$ (Fig 4.25). Each of the electrons within the annual region between the two cylinders gains an amount of momentum $q$ and energy $\Delta E$ given by
\[ \Delta E = \int \Delta \varepsilon \, dn = \frac{2 \pi n b \, db \, \Delta x}{8 \pi^{2} e_{0}^{2} m \beta v^{2}} \quad \text{(4.13-3)} \]

Since the volume of the annular region of length \( \Delta x \) between the two coaxial cylinders is \( 2 \pi n b \, db \), the number of electrons in this volume is \( dn = 2 \pi n b \, db \, \Delta x \). So the energy gained by all the electrons in the thickness \( \Delta x \) of the medium will be

\[ \Delta E = \int \Delta \varepsilon \, dn = \frac{2 \pi \varepsilon_{n} b \, \Delta x}{4 \pi^{2} e_{0}^{2} m \beta v^{2}} \frac{b_{\text{max}}}{b_{\text{min}}} \]

This is also the amount of energy lost by the incident particle in the thickness \( \Delta x \) of the substance. If \( N \) is the number of atoms per unit volume and \( Z \) is the atomic number of the medium, then \( n = NZ \). Hence the rate of energy loss per unit thickness of the substance is

\[ \frac{dE}{dx} = \frac{2 \pi \varepsilon_{n} \Delta x}{4 \pi^{2} e_{0}^{2} m \beta v^{2}} NZ \ln \frac{b_{\text{max}}}{b_{\text{min}}} \quad \text{(4.13-5)} \]

This rate of energy loss is known as the specific energy loss or the stopping power of the medium \( S(E) \). To determine \( S(E) \) from Eq. (4.13-5) it is necessary to know \( b_{\text{max}} \) and \( b_{\text{min}} \).

(i) Let us first calculate \( b_{\text{max}} \): If we take the time of collision \( \tau = b/v \), then since the electron velocity is small compared to the velocity of the incident ion, we can write

\[ \tau = \frac{b}{v} \ll T_e = \frac{2 \pi}{\omega_e} \]

where \( T_e = 2 \pi / \omega_e \) is the time period of revolution of the electron in its orbit, \( \omega_e \) being the angular velocity of the latter. The maximum permissible value of \( b \) corresponds to the condition

\[ \frac{b_{\text{max}}}{v} = \frac{1}{\omega_e} \]

or,

\[ b_{\text{max}} = \frac{v}{\omega_e} = \frac{\hbar v}{\hbar \omega_e} = \frac{\hbar v}{I} \quad \text{(4.13-6)} \]

where \( I = \hbar \omega_e \) is the mean ionization energy of the electrons in the atoms of the medium.

If the velocity of the ions is very high, so the relativistic effects have to be taken into account, then the Coulomb field of the ion becomes compressed in the direction of velocity of the ion. As a result, the perpendicular component of the field is increased by a factor

\[ y = (1 - \beta^{2})^{-1/2} \]

This shortens the duration of the impulse given to the electron so, that we get

\[ \tau = \frac{b}{v} (1 - \beta^{2})^{-1/2} \quad \text{(4.13-7)} \]

or,

\[ b_{\text{max}} = \frac{v}{\omega} = \frac{\hbar v}{\hbar \omega} = \frac{\hbar v}{I} \frac{1}{1 - \beta^{2}} \quad \text{(4.13-8)} \]

(ii) We now calculate \( b_{\text{min}} \):

We shall first estimate \( b_{\text{min}} \) by a method which is essentially classical. For this we calculate the maximum velocity transferred to an electron in the ion-electron collision.

Suppose an ion of mass \( M \) and velocity \( v \) collides elastically with an electron of mass \( m \) which is at rest. After the collision, the electron and the ion move in the directions making angles \( \theta \) and \( \phi \) respectively w.r.t. the incident direction with the velocities \( v_e \) and \( v_i \), respectively, as shown in Fig. 4.26. We consider the collision classically.

Applying the laws of conservation of energy and momentum, we get

\[ \frac{1}{2} M v^2 = \frac{1}{2} M v_i^2 + \frac{1}{2} m v_e^2 \]

or

\[ M v = M v_i \cos \phi + m v_e \cos \theta \]

or

\[ 0 = M v_i \sin \phi - m v_e \sin \theta \]

From Eqs. (4.13-10) and (4.13-11), we get by squaring and adding

\[ M^2 v_i^2 (\sin \phi)^2 + m^2 v_e^2 \sin^2 \theta = M^2 v^2 + m^2 v_e^2 - 2M v v_e \cos \theta \]

or

\[ v_i^2 = \frac{v^2 + \frac{m^2}{M}}{2} v_e^2 - 2 \left( \frac{m}{M} \right) v v_e \cos \theta \]

Again from Eq. (4.13-9), we have

\[ v_i^2 = v^2 - \left( \frac{m}{M} \right) v_e^2 \]

Comparing Eqs. (4.13-12) and (4.13-13) we get

\[ \frac{v^2 - \frac{m^2}{M}}{2} v_e^2 = \frac{v^2 + \frac{m^2}{M}}{2} v_e^2 - 2 \left( \frac{m}{M} \right) v v_e \cos \theta \]

or

\[ v_e \left( 1 + \frac{m}{M} \right) = 2 \frac{v \cos \theta}{m + M} \]

\[ \therefore \quad v_e = \frac{2 M v \cos \theta}{m + M} \quad \text{(4.12-14)} \]
Since the volume of the annular region of length $\Delta x$ between the two coaxial cylinders is $2 \pi b db \Delta x$, the number of electrons in this volume is $dn = 2 \pi b db \Delta x$. So the energy gained by all the electrons in the thickness $\Delta x$ of the medium will be

$$\Delta E = \int \Delta \varepsilon dn = 2 \pi e^4 n \Delta x / 8 \pi e_0^2 m b^2 v^2$$

Fig. 4.25. Electrons within an annular region of length $\Delta x$ between the two coaxial cylinders.

This is also the amount of energy lost by the incident particle in the thickness $\Delta x$ of the substance. If $N$ is the number of atoms per unit volume and $Z$ is the atomic number of the medium, then $n = N/Z$. Hence the rate of energy loss per unit thickness of the substance is

$$\frac{dE}{dx} = \frac{2 \pi e^4}{4 \pi e_0^2 m b^2 v^2} N \frac{b_{\text{max}}}{b_{\text{min}}}$$

This rate of energy loss is known as the specific energy loss or the stopping power of the medium $S(E)$. To determine $S(E)$ from Eq. (4.13-5), it is necessary to known $b_{\text{max}}$ and $b_{\text{min}}$.

(i) Let us first calculate $b_{\text{max}}$ : If we take the time of collision $\tau = b/v$, then since the electron velocity is small compared to the velocity of the incident ion, we can write

$$\tau = \frac{b}{v} = \frac{2 \pi}{\omega_e}$$

where $T_e = 2 \pi / \omega_e$ is the time period of revolution of the electron in its orbit, $\omega_e$ being the angular velocity of the latter. The maximum permissible value of $b$ corresponds to the condition

$$b_{\text{max}} = \frac{1}{\omega_e} \frac{v}{\omega_e} = \frac{\hbar v}{\omega_e}$$

or,

$$b_{\text{max}} = \frac{v}{\omega_e} = \frac{\hbar v}{\omega_e}$$

where $I = \hbar / \omega_e$ is the mean ionization energy of the electrons in the atom of the medium.

If the velocity of the ions is very high, so the relativistic effects have to be taken into account, then the Coulomb field of the ion becomes compressed in the direction of velocity of the ion. As a result, the perpendicular component of the field is increased by a factor

$$\gamma = (1 - \beta^2)^{-1/2}$$

This shortens the duration of the impulse given to the electron so, that we get

$$\tau = \frac{b}{v} = \frac{v}{\omega_e} \sqrt{1 - \beta^2}$$

or,

$$b_{\text{max}} = \frac{v}{\omega_e} \frac{1}{\sqrt{1 - \beta^2}} = \frac{\hbar v}{\omega_e} \sqrt{1 - \beta^2}$$

(ii) We now calculate $b_{\text{min}}$:

We shall first estimate $b_{\text{min}}$ by a method which is essentially classical. For this we calculate the maximum velocity transferred to an electron in the ion-electron collision.

Suppose ion of mass $M$ and velocity $v$ collides elastically with an electron of mass $m$ which is at rest. After the collision, the electron and the ion move in the directions making angles $\theta$ and $\Phi$ respectively w.r.t. the incident direction with the velocities $v_e$ and $v_i$, respectively, as shown in Fig. 4.26. We consider the collision classically.

Applying the laws of conservation of energy and momentum, we get

$$\frac{1}{2} M v^2 = \frac{1}{2} M v_i^2 + \frac{1}{2} m v_e^2$$

$$M v = M v_i \cos \Phi + m v_e \cos \theta$$

where $0 = M v_i \sin \Phi - m v_e \sin \theta$.

From Eqs. (4.13-10) and (4.13-11), we get by squaring and adding

$$M^2 v_e^2 = (M v - m v_e \cos \Phi)^2 + (m v_e \sin \theta)^2$$

or,

$$v_e^2 = v^2 + \left( \frac{m}{M} \right) v_e^2 - 2 M m v v_e \cos \theta$$

Again from Eq. (4.13-9), we have

$$v_i^2 = v^2 - \left( \frac{m}{M} \right) v_e^2$$

Comparing Eqs. (4.13-12) and (4.13-13) we get

$$v_e^2 = v^2 - \left( \frac{m}{M} \right) v_e^2 - 2 M m v v_e \cos \theta$$

or

$$v_e^2 \left( 1 + \frac{m}{M} \right) = 2 v v \cos \theta$$

...(4.13-14)
The maximum electron velocity corresponds to $\theta = 0$ for which

$$v_e = v_{\text{max}} = \frac{2Mv}{m + M} \quad \text{...(3.13-15)}$$

Since $M >> m$, we get

$$v_{\text{max}} = 2v \quad \text{...(4.13-16)}$$

Now the energy transferred to the electron in a collision for an impact parameter $b$ as given by Eq. (4.13-3), is

$$\Delta E = \frac{Z e^4}{8\pi^2 \varepsilon_0^2 m b^2 v^2}$$

For maximum velocity transferred, this must be equated to

$$\Delta E = \frac{1}{2} m (2v)^2 = 2mv^2 \quad \text{...(4.13-17)}$$

This gives

$$b_{\text{min}} = \frac{\xi}{Z e^2}{4\pi \varepsilon_0 m v^2} \quad \text{...(4.13-18)}$$

This is non-relativistic. With relativity correction, this becomes

$$b_{\text{min}} = \frac{\xi}{Z e^2}{4\pi \varepsilon_0 m v^2} \quad \text{...(4.13-18a)}$$

We now estimate $b_{\text{min}}$ from quantum mechanical considerations. If the assumption (b) made at the beginning of this section is valid, then we may state that $b$ must be greater than the de Broglie wavelength $\lambda = h/q$ of the electron where $q$ is the momentum received by the latter due to the impact.

Now in a frame of reference in which the incident ion is at rest (which is almost the same as the C-system) the velocity of the electron is equal to $v$ in magnitude. Hence $\lambda = h/mv$, so that

$$b_{\text{min}} = \frac{\xi}{Z e^2}{\frac{h}{mv}} \quad \text{...(4.13-19)}$$

With relativity correction this becomes

$$b_{\text{min}} = \frac{\xi}{Z e^2}{\frac{\sqrt{1 - \beta^2}}{mv}} \quad \text{...(4.13-19a)}$$

The larger of $(b_{\text{min}})_{\text{cl}}$ and $(b_{\text{min}})_{\text{qm}}$ has to be used in the integral (4.13-4). To see which of these is larger we estimate the ratio

$$\frac{b_{\text{min}}}{b_{\text{min}}^{\text{qm}}} = \frac{\xi}{Z e^2}{\frac{\frac{h}{mv}}{\frac{4\pi \varepsilon_0 m v^2}{\sqrt{\frac{1 - \beta^2}{c}}}}} = \frac{4\pi \varepsilon_0 \xi c}{Z e^2 \sqrt{1 - \beta^2}}$$

where $\alpha = e^2/4\pi \varepsilon_0 c = 1/137$ is the fine-structure constant. For $\beta = 0.1$ (which is the case for protons of 10 MeV energy), we get the above ratio as 13.7/2. For not too heavy ions, e.g., $p, d, \alpha$ etc., we must then use $(b_{\text{min}})_{\text{qm}}$ as the lower limit in the integral (4.13-4).

We then have, using Eqs. (4.13 - 8) and (4.13 - 19a)
the screening distance becomes small and the polarization effect becomes important. It is usually difficult to make precise theoretical calculation of the polarization effect. So the effect is usually estimated from empirical data.

Fig. 4.27: Variation of ionization loss with energy.

At very low energies the $S(E)$ vs. $E$ graph shows a maximum at B as seen in the figure. At still lower velocities, there is a decrease of $S(E)$ with decrease of velocity (the portion BA of the graph) which is not predicted by Eq. (4.13-22). The decrease is mainly due to the capture of electrons by the incoming ion which reduces its ionizing power. For example an $\alpha$-particle (He$^{++}$) may change to He$^+$ due to capture of an electron, which may then capture another electron and becomes a neutral helium atom. These may then lose electrons and become He$^+$ and He$^{++}$. Such capture and loss of electrons takes place many times near the end of the path of the particles when its velocity is quite low ($\sim 10^3$ times its last centimetre of the path).

The energy loss process discussed above is known as the ionization loss or collision loss. If $\omega$ is the average energy loss per collision and $n(E)$ is the number of ion-pairs produced per unit path within the medium, then we can write

$$S(E) = \frac{dE}{dx} = \omega n(E) \quad \ldots \quad (4.13-25)$$

$\omega$ is independent of the velocity of the incident particle. It depends on the nature of the medium. It may be noted that the incident particle not only ionizes the atoms in the medium, but also excites them in some collisions. Further, the electrons liberated in ionizing collisions have some kinetic energy. The ionization may also take place from different atomic orbits. For all these reasons, $\omega$ is usually much higher than the first ionization potential of the atoms of the medium. In air $\omega$ has a value of about 35 eV.

4.14 Theoretical calculation of the range of heavy charged particles

In § 4.6 we saw that the $\alpha$-particles of the same initial energy travel through the same distance in matter, which is known as their range. This is also true for other heavy charged particles, like the deuteron and the proton.

Eq. (4.13-22) can be used to calculate the range of the incident ion in the medium:

$$R = \int_0^R dx = \int_0^E \frac{dE}{dv} = \int_0^{E_0} \frac{dE}{S(E)}$$

Here $E_0$ is the initial energy of the incident particle. When the particle is at the end of its path of travel, i.e., for $x=R$, $E=0$. Since $E = Mv^2/2$, $dE = Mv dv$, so that we have

$$S(E) = -\frac{dE}{dv} = -Mv \frac{dz^2}{dz^2 f(v)}$$

from Eq. (4.13-22). Here $f(v)$ is a slowly varying function of velocity $v$. If we take a mean value of $f(v)$, then we can write

$$R = \int_0^{E_0} \frac{v^2 dE}{z^2 f(v)} = \frac{1}{z^2} \langle f(v) \rangle = \int_0^{\infty} Mv^3 dv$$

$$= \frac{M}{z^2} \langle f(v) \rangle \approx \frac{4}{z^2}$$

...(4.14-1)

Here $\langle f(v) \rangle$ is the mean value of $f(v)$ within the whole path of travel of the particle. Eq. (4.14-1) shows that the range $R$ of the particle is proportional to $v_0^4$, where $v_0$ is the initial velocity. Actually it is found that $R \propto v_0^3$ (Geiger's law) in certain limited energy ranges (see § 4.7).

As an example of the application of Eq. (4.14-1), the ratios of the ranges of $\alpha$-particles, deuterons and protons of the same initial velocity will be

$$R_\alpha : R_d : R_p = \frac{M}{z^2}_\alpha : \frac{M}{z^2}_d : \frac{M}{z^2}_p = 1 : 2 : 1$$

Here we have taken the masses $M$ of the particles to be equal to their mass numbers. Evidently the ratios of the initial energies of the above particles for the same $v_0$ are

$$E_\alpha : E_d : E_p = 4 : 2 : 1$$

As stated above, theoretically derived range–velocity relation does not hold in all energy regions (see also § 4.7). Empirical relationships based on the measured values of ranges for different particles in extended energy regions have been proposed by various workers.

The specific energy loss formula (4.13-22) holds when the velocity of the particle is large compared to $2v_0$ where $v_0 = 2.2 \times 10^6$ m/s is the velocity of the electron in the Bohr orbit. This limit corresponds to $0.1$ $MeV$ which is 100 keV for protons and 1.6 MeV for $\alpha$-particles. Because of the continual change of the charge of the particle due to the capture and loss processes mentioned above, we have to use an effective
In nuclear physics, the charge ($z^*$) for the particle at low energies in place of $z$ has been estimated by Bohr and is given by

$$z^* = z^{1/3} \frac{v}{v_0}$$

when the particle velocity is $v$.

As stated above, for $v >> 2z v_0$, $z^* = z$.

In general, the stopping power formula (4.13-22) is modified by replacing $z^2$ in it by $<z^2>$ which is the mean squared charge of the particle given by $<z^2> = a z^2$ where $a$ is a constant depending upon the ion velocity.

Semi-empirical expression for $a$ has been proposed with the help of which the behaviour of the stopping power formula at low energies can be satisfactorily explained.

At extremely low velocities ($E < 10$ keV per nucleon) the electron capture by the ion is almost complete so that the collisions are between the neutralized ions and the orbital electrons in the atoms of the medium. These are mostly elastic in the C.M. frame. There is also some contribution to the energy loss due to Rutherford scattering from the nuclei at such low energies which becomes important for heavy ions.

### 4.15 Straggling of range

In § 4.6 we saw that though the $\alpha$-particles of the same initial energy have more or less the same range in matter, there is a small spread in the values of the ranges about a mean, which is known as the straggling of the range. Fig. 4.11 showing the cloud chamber photographs of the $\alpha$-particle tracks clearly shows the evidence of straggling, the different tracks in the same $\alpha$-group having slightly different lengths. If the ranges of all the $\alpha$-particles of the same energy has the same range, the $\alpha$-intensity vs. distance curve in Fig. 4.14 would drop abruptly to zero at $x = R$. Similar behaviour of the ionization density vs. distance curve in Fig. 4.10 would also be expected. The gradual sloping of these curves near the end is due to the straggling effect.

The straggling of the range of heavy charged particles occurs mainly due to two reasons: (i) firstly there is a statistical fluctuation in the number of collisions suffered by the different particles about a mean value in travelling through a given distance. Thus the different $\alpha$-particles would have to move through different distances in order to lose their entire initial energy; (ii) secondly the energy lost per collision also has a statistical fluctuation about a mean value.

Other factors responsible for straggling include the multiple scattering of the particles during collisions (which is particularly important in producing the straggling of the range of the $\beta$-particles), density inhomogeneity of the absorbing medium and the capture and loss of electrons by the moving particle.

If we assume a normal distribution for the straggling of the range, then the number of particles with ranges between $x$ and $x + dx$ is given by

$$dn = \frac{n_0}{\sigma_R \sqrt{2 \pi}} \exp \left\{ - \frac{(x - <R>)^2}{2\sigma_R^2} \right\} dx \quad (4.15-1)$$

where $2 \alpha_R^2$ is called the straggling parameter and $<R>$ is the mean range. $\sigma_R$ is the standard deviation in the range given by

$$\sigma_R^2 = <R^2> - <R>^2$$

$$\frac{\sigma_R^2}{<R>} = \frac{2}{\sigma_0^2} \int_0^E \left( \frac{dE}{dx} \right)^3 dE \quad (4.15-2)$$

which gives

$$\frac{\sigma_R^2}{<R>} = 2 \frac{m}{M} \frac{1}{\ln (2m^2/v^2)} \quad (4.15-3)$$

For 5 MeV $\alpha$-particles, we get $\sigma_R/<R> = 0.95\%$ while for protons of the same velocity ($E = 1.25$ MeV), $\sigma_R/<R> = 1.9\%$ in air.

Integration of Eq. (4.15-1) gives the number-distance curve of Fig. 4.14. At $x = <R>$, the exponential factor is unity so that the slope of the number-distance curve at this point is

$$\frac{1}{n_0} \frac{dn}{dx} = \frac{l}{\sigma_R \sqrt{2\pi}} \quad (4.15-4)$$

where $x = <R>$, is a point of inflection, as can be seen by differentiating Eq. (4.15-1). From the properties of the normal distribution, it is known that the number of particles with ranges less and greater respectively than the mean range $<R>$ is half the total number. This is illustrated in Fig. 4.28 which shows the end portion 'a' of the number vs. distance graph of Fig. 4.14. Also shown in the same figure is the differential of the above graph 'b', which has a Gaussian (normal) shape. The half-width of this differential range curve at $(1/e)^{th}$ of its maximum is $\sigma_R/\sqrt{2}$.

The symmetry of this graph about $x = <R>$ shows that half of the particles have ranges greater than $<R>$, while the other half have ranges less than $<R>$.

Assuming the decreasing portion ABC of the number vs. distance curve to be linear, with a slope equal to that at $x = <R>$ as given by Eq. (4.15-4), we can find the extrapolated range $R_{ex}$ which is the value of the abscissa at which the straight line ABC intersects the $x$-axis. Since the ordinate at $<R>$ is half the original number of particles, we can write the slope as
\[
\frac{1}{\sigma_R \sqrt{2\pi}} = \frac{0.5}{R_{ex} - <R>}
\]

or
\[
R_{ex} - <R> = \frac{\sigma_R \sqrt{2\pi}}{2} = 3.842 + 0.055
\]

\[
= 3.897 \text{ cm}
\]

This shows that \( R_{ex} \) is about 1.4\% greater than the mean range \(<R>\) which is a measure of the straggling of the range.

**References**


**Problems**

1. \( ^{210}\text{Po} \alpha\)-particles have kinetic energy of 8.776 MeV. Assuming the mass of the \( \alpha\)-particles to be \(6.67 \times 10^{-21}\) kg, calculate their velocity.

\[
(2.05 \times 10^7 \text{ m/s})
\]

2. \( ^{216}\text{Po} \alpha\)-particles having a kinetic energy of 5.3 MeV, are subjected to a magnetic field of 1 T. What is the radius of curvature of their tracks? The charge on the \( \alpha\)-particles is \(3.2 \times 10^{-19}\) C.

\[
(0.333 \text{ m})
\]

3. Alpha particles from \( ^{210}\text{Po} \) source, having the kinetic energy of 5.3 MeV enter the region between two parallel metallic plates separated by 5 mm. A p.d. of 5000 volts is applied between the plates. The \( \alpha\)-particles coming out through a narrow slit on the far side of the plates (of linear dimension 0.5 m) reach a photographic plate, kept at a distance of 0.5 m behind the slit. The deflection of the \( \alpha\)-particles from the central spot on the plates is found to be 2.4 cm. The electric field is then removed and a magnetic induction field of 0.1 T is applied to the \( \alpha\)-particles perpendicularly under the same geometry. The deflection on the photographic plate from the central spot is found to be 7.6 cm. Find the specific charge of the \( \alpha\)-particles from the above data.

\[
(4.813 \times 10^7 \text{ C/kg})
\]

4. Using the data in Probs. 1 and 2, calculate the \( \alpha\)-disintegration energies for \( ^{212}\text{Po} \) and \( ^{216}\text{Po} \). Calculate the kinetic energies and velocities of the recoil nuclei in the two cases. What percentages of the disintegration energies in each case are shared by \( \alpha\)-particle and the recoil nucleus?

5. \( ^{226}\text{Ra} \) which decays by \( \alpha\)-emission into \( ^{222}\text{Rn} \), has an atomic mass of 226.025436 u. The \( \alpha\)-disintegration energy is \(Q = 4.863\) MeV. Assuming the mass of the \( \text{He} \) atom to be 4.002603 u, calculate the atomic mass of \( ^{222}\text{Rn} \), given \( 1 \text{ u} = 931.5 \text{ MeV} \).

\[
(222.017604 \text{ u})
\]

6. The mean range of \( ^{210}\text{Po} \) \( \alpha\)-particles (\( E = 5.3 \) MeV) in air at S.T.P. is 0.03842 cm. Find its range in aluminium \( (A = 27, \rho = 2700 \text{ kg/m}^3 \) both in metre and \( n \text{ kg/m}^2 \). \( (2.36 \times 10^{-2}\text{ m}; 6.3 \text{ kg/m}^2) \)

7. Using Geiger's law estimate the range of \( ^{220}\text{Rn} \) \( \alpha\)-particles \( (E = 6.2823 \text{ MeV}) \) using the data given in Prob. 6.

\[
(0.0496 \text{ m})
\]

8. Integrate Eq. (4.10-3) to obtain Eq. (4.10-4) (Hint : Make the substitution \( r = b \cos^2 \theta \)).

9. Calculate the barrier penetration factor \( G \) for the \( \alpha\)-disintegration energy 4.25 MeV. Hence calculate the half-life of the disintegration. Compare with the actual value and notice the difference (Hint: Calculate \( R \) from the formula \( R = 1.4 \times 10^{-15} A^{1/3} \). Use the actual value of the term within the second bracket in Eq. (4.10-4)).

\[
(1.32 \times 10^{12} \text{ y})
\]

10. Protons of kinetic energy 1 MeV have the stopping power of 175 keV per mg per cm\(^2\) in aluminium. At what energies will muons, deuterons and \( \alpha\)-particles have the same stopping power in aluminium? Neglect the variation of the log term in Eq. (4.12-22). (Moons are particles of electronic charge about 207 times heavier than the electrons).

\[
(0.113; 2; 16 \text{ MeV})
\]

An \( \alpha\)-particle has a range of 250 micrometeres in a nuclear emulsion plate. What would be the ranges in the same emulsion of a \( ^{3}\text{He} \) nucleus and of a \( ^{3}\text{H} \) nucleus, if they have the same initial velocity as the \( \alpha\)-particle?

\[
(187.5 \mu\text{m}; 750 \mu\text{m})
\]

2. Estimate the value of \( \sigma_R / <R> \) for 5 MeV \( \alpha\)-particles in air. Use Geiger's law for finding the mean range \(<R>\). \( \sigma_R \) is the standard deviation in range. Hence find \( \sigma_R / <R> \) for deuterons of the same initial velocity.

\[
(0.9\%; 1.27\%)
\]
5

Beta Particles and Beta Activity

5.1 Determination of the specific charge of the beta particles

We have seen that the $\beta$-particles emitted by radioactive substances are very high energy electrons. Madame Curie was the first to show that the $\beta$-particles were a type of negatively charged particles and were much more penetrating than the $\alpha$-particles. Later, Becquerel observed that these $\beta$-particles were deflected much more than the $\alpha$-particles in an electric field. This proved that the $\beta$-particles are much lighter than the $\alpha$-particles. Becquerel obtained a rough estimate of the $e/m$ of the $\beta$-particles, which showed that they were electrons. He also observed that unlike the $\alpha$-particles, the $\beta$-particles emitted by radioactive substances were mono-energetic.

Kaufmann was the first to determine the $e/m$ of the $\beta$-particles accurately (1901). His apparatus is shown in Fig. 5.1. Within the vacuum chamber $A$, a small amount of radium was kept as the source ($Q$) of $\beta$-particles. These passed through the narrow gap between the two metal plates $P_1$ and $P_2$ which were kept at a potential difference of several thousand volts. The $\beta$-particles emerging at the far side of the plates $P_1$ and $P_2$ passed through the diaphragm $D$ and fell on the photographic plate $P$. The whole apparatus was placed between the pole pieces $N$, $S$ of a magnet. The magnetic field was parallel to the plane of the paper and perpendicular to the direction of motion of the $\beta$-particles.

If $m$, $e$ and $v$ be respectively the mass, charge and velocity of the $\beta$-particles, then due to the action of the magnetic induction field $B$ in the plane of the paper, the $\beta$-particles are deflected perpendicular to the plane of the paper, while the electric deflection due to the electric field $E$ between $P_1$ and $P_2$ is in the plane of the paper. The magnetic deflection is proportional to $e/m$ while the electric deflection is proportional to $e/mv^2$ (see Ch. II in Vol. I). Since the electric and magnetic deflections are at right angles to each other, the method is similar to that of determining the specific charge of the positive rays developed by J. J. Thomson. As discussed in Ch. II of Vol. I, the ions of the same specific charge, but different velocities, are focussed along a parabolic line. In Kaufmann's experiment also, the $\beta$-particles from a given source were found to be focussed along a parabola, which showed that they were emitted with different velocities from the source. By reversing the magnetic field, another parabola was found on the other side of the axial line (see Fig. 5.2). From the distance between the similar points on the two parabolas on the two sides of the axis, $e/m$ could be determined. Kaufmann found that though $e/m$ for the slower $\beta$-particles had the same value, the $e/m$ for the faster $\beta$-particles seemed to depend on their velocity. Higher the velocity, lower was the value of $e/m$. He also observed that the line along which the faster $\beta$-particles were focussed deviated from the parabolic line along which the slower $\beta$-particles were focussed (dashed line in Fig. 5.2).

These observations of Kaufmann were in agreement with the predictions of Einstein's Special Theory of Relativity. According to this, the mass $m$ of a particle of velocity $v$ is greater than its rest mass $m_0$ and is given by (see Ch. XV in Vol. I)

$$m = \frac{m_0}{\sqrt{1 - \beta^2}} \quad \cdots \quad (5.1-1)$$

where $\beta = v/c$. In the case of the lighter particle like the electron, at energies of the order of an MeV (10$^6$ eV), the velocity $v$ of the particle becomes comparable to the velocity of light ($c$). Hence for such particles, the relativistic increase of mass becomes appreciable; for heavier particles, like the proton or the $\alpha$-particle of comparable energies $v < c$, so that the mass-increase is negligible. As an example for electrons of 1 MeV energy $\beta = 0.941$ which gives $m = 3m_0$. For protons of the same energy, $\beta = 0.0462$ so that the mass-increase is negligible.

As the velocity of the particle increases, $m$ becomes higher and hence $e/m$ becomes smaller, in agreement with the observations of Kaufmann. However, in Kaufmann's experiment, the increase in the mass could not be measured accurately. Hence it could not be ascertained whether the increase was according to Einstein's formula.

The deviation of the focussing line for the high velocity electrons from the parabola can also be understood from the Special Theory of Relativity. In the theory of Thomson's parabola method of determining
the specific charge of the positive rays discussed in Vol. I, it is assumed that the parabolic relationship between the coordinates $x$ and $y$ of the focussing points of the ions of different velocities depends on their specific charge. If, for some reasons, the mass of the ion changes, this mathematical relationship will no longer hold. Since the mass of the high velocity electrons changes with velocity, their focussing line will deviate from the parabolic line.

5.2 Bucherer's experiment

A. H. Bucherer, in Germany, devised a very accurate method of determining the $e/m$ of the $\beta$-particles. His experiment also provided a precise test for the verification of Einstein's relationship (Eq. 5.1-1) for the variation of the mass with velocity.

![Fig. 5.3. Bucherer's method for determining $e/m$ for $\beta$-particles. (a) Path of the $\beta$-particles at right angles to the magnetic field. (b) Path of the $\beta$-particles parallel to the magnetic field.](image)

The experimental arrangement is shown in Fig. 5.3. A small amount of radium fluoride was used as the source (S) of the $\beta$-particles, which was placed between two parallel circular glass plates $P_1$, $P_2$ coated with silver on the inside faces, which acted as the two plates of a parallel plate condenser. The plates were 8 cm in diameter with a gap of 0.25 mm between them. The source S was placed at the centre of the plate system, midway between them. F is a photographic film, wound in the shape of a cylinder, surrounding the plates $P_1$, $P_2$ coaxially, 5 cm away from their periphery. The whole apparatus was enclosed in a chamber which was kept under very high vacuum.

A large potential difference was applied between $P_1$ and $P_2$ to produce an electric field $X$ in the vertical direction. In addition, a magnetic field was applied parallel to the plates, so that the electric and magnetic deflections were parallel. Since the gap between $P_1$ and $P_2$ was very small, only the $\beta$-particles emitted parallel to their surfaces could come out of the region between them to reach the photographic film F. The electrical force on the $\beta$-particles of charge $e$ due to the field $X$ was $F_e = Xe$. Due to this, the $\beta$-particles emitted from the source were deflected downwards in the absence of any magnetic field and hit the lower plate $P_2$ so that they could not emerge from the region between the plates. The direction of the magnetic induction field $B$ was so arranged that the magnetic force $F_m = Bev$ cancelled the electric force $F_e$ which allowed the $\beta$-particles emitted from S parallel to $P_1$, $P_2$ to emerge from the region between the plates.

Since the magnetic field was in a definite direction parallel to the plates, the $\beta$-particles emitted in different directions from S subtended different angles $\theta$ with $B$ (see Fig. 5.4).

If $v$ be the velocity of the $\beta$-particles emitted at the angle $\theta$ w.r.t. $B$, the magnetic force acting on it is $F_m = Bev \sin \theta$.

If the electric and the magnetic forces balance, then we can write $Bev \sin \theta = Xe$.

$\therefore \quad v = \frac{X}{B \sin \theta} \quad \ldots (5.2-1)$

For definite values of $X$ and $B$ only those $\beta$-particles emitted in the direction $\theta$ can emerge from the condenser which have velocity determined by Eq. (5.2-1). After their emergence from the condenser, only the magnetic force acts on them; the electric force does not act beyond the condenser plates. So the $\beta$-particles are deflected upwards or downwards and travel towards F along spiralling paths. Only those particles for which $\theta = \pi/2$ travel along paths which are circular arcs. For them, we can equate the magnetic force to the centripetal force, which gives $Bev = \frac{mv^2}{r} \quad \ldots (5.2-2)$

These particles have the maximum deflection.

If $l$ be the distance from the periphery of the condenser plates to the photographic film F and the deflection of the particles coming out at $\theta = \pi/2$ be $d$, then from Fig. 5.5, we get for $\theta = \pi/2$:

$$(2r - d) \cdot d = l^2 \quad \ldots (5.2-3)$$

or,

$$r = \frac{l^2 + d^2}{2d} \quad \ldots (5.2-3')$$

So from Eq. (5.2-2) we get, using Eq. (5.2-1)
By measuring the deflection $d$ on the film $F$, $e/m$ can be found.

In Bucherer’s experiment, the β-particles emitted from the source in different directions were allowed to fall on the film, keeping $X$ and $B$ constant. After exposing the film for some time, the electric and magnetic fields were reversed and the film was exposed again for some time as before. On developing the film, two symmetric dark lines were obtained on either side of the axial line. These had the appearance as shown in Fig. 5.6. The deflection was the maximum for $\theta = \pi / 2$, while for $\theta = 0$ and $\pi$, the deflections were minimum. In between, the deflections varied monotonously between the above two values.

In the actual arrangement of Bucherer’s experiment, the electric and magnetic fields were so adjusted that $X/B = c/2$ where $c$ is the velocity of light in vacuum. So according to Eq. (5.2-1), $\sin \theta = \frac{1}{2}$. Since the maximum possible value of $\beta$ is 1, this gives the minimum value $\theta_{\text{min}} = 30^\circ$. Thus the traces of the dark lines, instead of extending from $0^\circ$ to $180^\circ$, actually extended from $30^\circ$ to $150^\circ$ in Bucherer’s experiment.

It must be emphasized that the different points on the dark lines traced on the film were due to the β-particles of different velocities. If we consider the relativistic increase of the mass of the electron, given by Eq. (5.1-1), we get from Eq. (5.2-4)

$$e = \frac{2X d}{m} = \frac{1}{B^2 (P^2 + d^2)} \quad \ldots (5.2-5)$$

In Bucherer’s experiment, $\beta_{\text{max}}$ was 0.7 ($\gamma_{\text{max}} = 0.7c$). According to Eq. (5.1-1), the electron mass in this case was $7/5$ times its rest mass. Bucherer’s experiment established the validity of Einstein’s relation for the relativistic increase of mass with velocity. Eq. (5.2-5) was used to determine $e/m_0$ which agreed well with the values of the specific charge of electrons determined by other methods.

There were some sources of error in Bucherer’s experiment which were later corrected by other workers. These later measurements which were more accurate, have established the validity of the relativistic relation (5.1-1) unambiguously. In this connection the experiment of M. M. Rogers, A.W. McReynolds and F. T. Rogers is particularly noteworthy. They used the internal conversion electrons from a mixture of RaB and RaC in their experiment. The internal conversion electrons are mono-energetic (see Ch. VI). They first measured the momentum of the electrons with the help of a magnetic field. Later the electrons were deflected by a radial electric field. The mono-energetic electrons could cross the electric field and emerge on the other side, only if the field had a definite value. From the value of this critical field, $e/m$ for the high energy electrons could be measured.

### 5.3 Determination of the β energy

The experiments of Kaufmann and Bucherer showed that the β-particles from radioactive sources were emitted with a continuous distribution of energy.

The energy distribution of the β-particles can be determined with the help of a semi-circular focusing magnetic spectograph, briefly described in § 4.4. Such an apparatus designed by Rutherford and Robinson, is shown in Fig. 5.7. $S$ is a very thin wire coated with the radioactive substance and acts as the source of β-particles. The β-particles emitted from $S$ emerge through the slit $A$ as a slightly divergent beam and describe circular paths due to a homogeneous magnetic field, applied perpendicular to the plane of the diagram. The β-particles of a particular velocity in the slightly divergent beam, after describing different semi-circular paths, are all focussed at a definite point on the photographic plate $P$, as shown in the figure. Because of the short length of the slit $A$, the particles are actually focussed on the plate along a short line perpendicular to the plane of the paper. In the above case β-particles of different velocities are focussed along different focal lines, which are parallel to one another. The focussing action of the magnetic field is discussed in the § 5.4.

To prevent the β-particles from the source directly hitting the photographic plate, the latter is shielded by a lead plate $R$. The whole apparatus is kept under very high vacuum.

Since the β-particles are emitted from the source with a continuous distribution of velocities, the different focal lines mentioned above form a continuous spectrum. As the plate is developed, it is found to be continuously blackened from the end nearest to the slit to a definite farthest point, which shows that the β-particles are emitted with velocities and hence energies ranging from 0 up to a maximum (see Fig. 5.8). The intensity of blackening is different at different points, which indicates that the intensities of the β-particles of different velocities are different.

Instead of a photographic plate, a Geiger-Muller counter may be used for detecting and counting the number of β-particles of different velocities. By shifting the position of the counter along the line $AP$, β-particles of different velocities can be detected and counted. In this way the energy-distribution of the β-particles from a given radioactive...
The corresponding numbers of P-particles for each energy interval at \( E_p \) is determined with the natural radio-element RaE \((^{210}\text{Bi})\) is plotted against \( E_p \). In Fig. 5.8 the energy distribution of the \( \beta \)-particles emitted by the artificially radioactive substance \(^{198}\text{Au}\) \((Z=79)\). Notice that the number of \( \beta \)-particles with \( E_p = 0 \) is finite.

In Fig. 5.9 is shown the momentum distribution of the \( \beta \)-particles emitted by the artificially radioactive substance \(^{198}\text{Au}\) \((Z=79)\). The abscessa in this graph is the product \((Br)\) which is a measure of the momentum \( p \) of the \( \beta \)-particles \((\text{see Eq. 5.3-1)}\). As in the other case, the number of \( \beta \)-particles vary continuously with \( Br \). However at some definite values of \( Br \), a number of sharp peaks are observed, which represent groups of mono-energetic electrons superimposed on the continuous distribution.

These are known as internal conversion electrons or simply conversion electrons. Rutherford and his coworkers showed that these secondary electrons were not emitted from the \( \beta \)-disintegrating nuclei. Their origin will be explained in § 6.11.

In later years more improved \( \beta \)-ray spectrometers have been devised. For example, K. Siegbahn of Sweden devised a long focus magnetic lens spectrometer. The instrument consists of a solenoid enclosed within a coaxial soft iron cylinder. An electric current in the solenoid produces an axial magnetic field. The solenoid encloses a coaxial vacuum chamber at one end of which a \( \beta \)-source is mounted axially. A small G-M counter placed axially at the other end counts the \( \beta \)-particles of different energies focussed into it by varying the axial magnetic field. The whole arrangement acts as a magnetic lens (see § 5.4). To prevent the \( \beta \)-particles emitted by the source from entering the counter directly, a lead screen is placed between the source and the counter. The \( \beta \)-particles emitted by the source within a small angular range at a definite angle with the axis, proceed along a helical path and are focussed on the thin mica window of the counter. The focussing field depends on the velocity of the \( \beta \)-particles, so that by changing the magnetic field, \( \beta \)-particles of different velocities can be made to enter the counter, with the help of which their numbers are counted.

It is possible to determine the energy distribution of the \( \beta \)-particles very accurately with this type of spectrometer which has a fairly high resolving power.

Fig. 5.11 shows a short-lens type magnetic spectrometer. The solenoid \( M \), acting as the magnetic lens, has linear dimensions small compared to the distance between the source \( R \) and the counter. Hence the name short lens spectrometer (see § 5.4).

5.4 Focussing of charged particles by a magnetic field

There are a number of important parameters which determine the performance of a magnetic spectrometer, e.g., their resolution, transmission factor etc. As in optics, the resolution determines how well...
If \( \alpha \) be the radius of the two paths we get. If \( OBMQ \) intersects the line \( OP \) at \( Q \). Since \( OBM \) is a semi-circle, the amplitude \( OBMQ \), which is a circular arc of which \( OM \) is a diameter. This intersects the line \( OP \) at \( P \). The focal action of the magnetic spectrometer starts from \( O \) in a direction perpendicular to the plane of the spectrometer, this should be as high as possible. If the two groups are not well-resolved, parts of each group will enter the G-M counter for the same magnetic field and hence the shape of the energy distribution obtained with the instrument will not faithfully reproduce the actual distribution. The transmission factor of the instrument measures the fraction of the total number of particles emitted from the source within a particular energy range which enters the counter. This quantity is of special importance in the case of very weak sources.

As we have seen, for mono-energetic electrons, a line is produced on the photographic plate. This line has a finite width, depending on the width of the slit through which the particles enter the spectrometer field and the divergence of the beam at entry. If a G-M counter is used as a detector, then variation of the magnetic field gives a sharp peak in the counting rate for mono-energetic electrons. For the reasons stated above, as also due to the finite width of the slit in front of the detector, the peak has a certain finite width.

Resolution of the instrument is given by \( \Delta E/E \) or \( \Delta p/p \) where \( \Delta E \) is the full width of the 'line' (i.e., the peak) at half maximum (FWHM) at the mean energy of the peak. \( \Delta p \) is similarly the momentum width.

Consider the simplest of the magnetic spectrometers which is the semi-circular focussing type. Suppose a system of slits limits the beam in the angular interval \( \alpha \) [see Fig. 5.10]. A \( \beta \)-particle from a point source \( O \), starting in a direction perpendicular to the line \( OP \), describes the semi-circular path \( OAP \) in a homogeneous magnetic field perpendicular to the plane of the paper. This intersects the line \( OP \) at \( P \). Another \( \beta \)-particle of the same energy starts from \( O \) in a direction perpendicular to \( OM \), which makes an angle \( \alpha \) with \( OP \) and describes the path \( OBMQ \), which is a circular arc of which \( OM \) is a diameter. The path \( OBMQ \) intersects the line \( OP \) at \( Q \). Since \( OBM \) is a semi-circle, the angle \( OQM \) is a right angle. If \( R \) be the radius of the two paths we get

\[
OQ = OM \cos \alpha = 2R \cos \alpha
\]

\[
OP = 2R
\]

Hence the distance between the points of incidence of the two \( \beta \)-particles on the line \( OP \) is

\[
PQ = OP - OQ = 2R(1 - \cos \alpha)
\]

\[
= 2R \left(1 - 1 + \frac{\alpha^2}{2} - \frac{\alpha^4}{24} \ldots \right)
\]

\[
= R \alpha^2
\]

If \( \alpha \) is small. Thus to a first order of approximation, we can neglect the distance between \( P \) and \( Q \) if \( \alpha \) is very small, which means that the \( \beta \)-particles with a small angular divergence \( \alpha \) at entry are all focused at the same point. This is known as the first order focusing. Actually, as we have seen above, the \( \beta \)-particles are not all focused at a single point, so that the \( \beta \)-ray line has a finite width, even when they are mono-energetic. If \( s \) is the linear width on a photographic plate, then assuming a point source from which the emitted \( \beta \)-particles are limited to the angular interval \( \alpha \) by a slit, we have for a definite momentum \( p \) and a given field \( B \) (see Eq. 5.4-1)

\[
s = 2R \left(1 - \cos \alpha\right) = R \alpha^2
\]

and the momentum resolution is

\[
\frac{\Delta p}{p} = \frac{s}{2R} = \frac{\alpha^2}{2R}
\]

The solid angle subtended by the entrance slit at the source is

\[
\Delta \Omega = 2 \pi \left(1 - \cos \alpha\right)
\]

so that the transmission factor is

\[
T = \frac{\Delta \Omega}{4 \pi} = \frac{2 \pi \left(1 - \cos \alpha\right)}{4 \pi} = \frac{1}{2} \frac{\alpha^2}{2} = \frac{\alpha^2}{4}
\]

Typical values are \( \Delta R/R \sim 1\% \) and \( T \sim 0.1\% \).

The uncertainty \( \delta E \) in the absolute value of the energy, as measured by the spectrometer, depends on the precision with which the magnetic field can be measured. It also depends on the uniformity of the field.

Photographic plate is used as the detector usually when a permanent magnet is used, because of the long exposure needed, during which the field must remain absolutely steady. More usual practice is to use an electro-magnet and a counter at a fixed position and then vary the magnetic field to bring the \( \beta \)-particles of different energies to enter the counter. By plotting \( N(p)/p \) against \( B \) we get the momentum spectrum.

We have also discussed about magnetic lens spectrometers of various types (long focus or short focus) which are in more common use now-a-days. In this, the path of a particle emitted from the source \( S \) at an angle \( \alpha \) with the axis of the solenoid is a helix, the magnetic field being axial. After describing one full turn (or any integral number of turns), the helix cuts the axis again at the focus \( F \), where the counter is located. Fig. 5.11(a) shows the projection of the helical path in the plane of the paper.

If \( \nu \) be the velocity of the \( \beta \)-particles, its component perpendicular to the axis is \( \nu \sin \alpha \), so that we can write

\[
Be \sin \alpha = \frac{m(\nu \sin \alpha)^2}{R}
\]

or,

\[
R = \frac{mv \sin \alpha}{Be}
\]

Hence the angular velocity about the axis is

\[
\omega = \frac{\nu \sin \alpha}{R} = \frac{Be}{m}
\]
and the time period is

\[ T = \frac{2\pi m}{\omega} = \frac{2\pi m}{Be} \]

...(5.4-1)

The above instruments are all single focusing type which means that they form line images of point-sources. Double focusing instruments have also been devised which form point-images.

The principle of focusing of charged particles by a magnetic field finds important applications in the design of mass spectrometers (see Ch. VIII).

5.5 Energies of \( \beta \)-decay

Three different types of \( \beta \)-decay are observed, viz., \( \beta^- \) decay, \( \beta^+ \) decay and orbital electron capture. In all the three processes, the mass number \( A \) of the parent nucleus remains unchanged. Only the atomic number \( Z \) changes by one unit. Since \( A = Z + N \), where \( N \) is the neutron number, this means that the total number of neutrons and protons in the nucleus remains unchanged after \( \beta \)-decay. In \( \beta^- \) decay, \( Z \) is increased by one unit to \( (Z + 1) \) so that \( N \) is decreased to \( (N - 1) \). This happens because one neutron in the nucleus is transformed into a proton by \( \beta^- \) decay.

Many artificially produced radioactive nuclei are found to undergo \( \beta^+ \) decay. \( \beta^+ \) or the positron is the anti-particle of the electron. It has the same mass as the electron, but has equal and opposite charge (+e). When a nucleus undergoes \( \beta^+ \) decay it loses one unit of positive charge, so that \( Z \) is decreased to \( (Z - 1) \) and \( N \) increased to \( (N + 1) \). This happens because one proton in the nucleus is transformed into a neutron.

Finally, in orbital electron capture process, which is observed amongst many artificially produced radioactive nuclei, a proton within the nucleus captures an orbital electron (mostly the K-electron, which is nearest to the nucleus) and is transformed into a neutron. Thus, as in \( \beta^- \) decay, \( Z \) is reduced to \( (Z - 1) \) and \( N \) increased to \( (N + 1) \) in orbital electron capture type of \( \beta \)-decay.

Consider the \( \beta^+ \) decay of the nucleus of an atom X of atomic mass \( M(A, Z) \) into the nucleus of the atom Y of atomic mass \( M(A, Z + 1) \). We can represent the process by the equation

\[ \frac{\gamma}{X} \rightarrow \frac{\gamma}{X+1} Y + \beta^- \]

Using Einstein’s relation \( E = mc^2 \) between the mass \( m \) and the energy \( E \) of a particle, we can write the disintegration energy in \( \beta^- \) decay as the difference between the nuclear masses \( M_x(A, Z) \) and \( M_x(A, Z + 1) \) of X and Y respectively minus the electronic mass \( m_e \), whole multiplied by \( c^2 \):

\[ Q_{\beta^-} = [M_x(A, Z) - M_x(A, Z + 1) - m_e]c^2 \]
Instead of the nuclear masses, if we take the atomic masses, we get
\[ Q_{p^+} = [M(A, Z) - Z m_e - M(A, Z - 1) + (Z + 1) m_e - m_e^2/c^2 - B_e) \]
\[ = [M(A, Z) - Z m_e - M(A, Z - 1) + (Z + 1) m_e - m_e^2/c^2 - B_e] \] (5.5-4)
If the masses are all expressed in energy units, the above equation becomes
\[ Q_{p^+} = M(A, Z) - M(A, Z + 1) \] (5.5-5)
In the above equation the contribution from the electronic binding energy of the atom has been neglected.

In the case of \( \beta^+ \) decay, we can write
\[ A^2X \rightarrow A^2+1Y + \beta^- \]
and
\[ Q_{p^+} = [M(A, Z) - Z m_e - M(A, Z - 1) - m_e^2/c^2 - B_e] \]
\[ = [M(A, Z) - Z m_e - M(A, Z - 1) + (Z + 1) m_e - m_e^2/c^2 - B_e] \]
Expressing the masses in energy units, we get as before
\[ Q_{p^+} = M(A, Z) - M(A, Z - 1) - 2m_e \] (5.5-6)
Lastly in the case of orbital electron capture process, we write
\[ A^2X + e^- \rightarrow A^2-1Y \]
Since the electron in its orbit has a binding energy \( B_e \) (say), this must be subtracted from the energy released in the electron-capture process, to get the disintegration energy \( Q_e \):
\[ Q_e = [M(A, Z) + m_e - M(A, Z - 1)] c^2 - B_e \]
\[ = [M(A, Z) - Z m_e - M(A, Z - 1) + (Z + 1) m_e - m_e^2/c^2 - B_e] \]
Expressing the masses in energy units, we then get
\[ Q_e = M(A, Z) - M(A, Z - 1) - B_e \] (5.5-6)
From Eq. (5.5-2), we see that \( Q_{p^+} > 0 \) if \( M(A, Z) > M(A, Z + 1) \)
Hence \( \beta^+ \) disintegration can take place if the mass of the parent atom \( A^2X \) is greater than that of the daughter atom \( A^2+1Y \).

On the other hand, Eq. (5.5-4) shows that \( Q_{p^+} > 0 \) if \( M(A, Z) > M(A, Z - 1) + 2m_e \). So \( \beta^+ \) disintegration is possible if the mass of the parent atom is greater than the sum of the masses of the product atom and twice the electronic mass. Since the electron has a rest energy of 0.511 MeV, this means that the mass of the parent atom must be greater than that of the daughter atom (both expressed in energy units) by an amount at least equal to 1.022 MeV.

Finally, in the case of electron capture, we get from Eq. (5.5-4) \( Q_e > 0 \) if \( M(A, Z - 1) > M(A, Z - 1) \). This means that electron capture is possible if the mass of the parent atom \( A^2X \) is greater than the mass of the daughter atom \( A^2-1Y \) by at least the electron binding energy.

If the electron binding energy \( B_e \) is small, as in the case of the lighter atoms, this condition reduces to \( M(A, Z) \) being simply greater than \( M(A, Z - 1) \).

Electron capture decay and \( \beta^+ \) decay can occur in one and the same nucleus, provided both the Eqs. (5.5-4) and (5.5-6) are satisfied. If Eq. (5.5-6) is satisfied, but not Eq. (5.5-4), then only the electron capture decay is possible, but not \( \beta^-\)decay. An example is the decay of \( ^7\text{Be} \) into \( ^7\text{Li} \):
\[ ^7\text{Be} + e^- \rightarrow ^7\text{Li} \]

This decay cannot take place by \( \beta^- \) emission, because the difference in the atomic masses of the two nuclei, which is 0.864 MeV is less than \( m_e c^2 = 1.022 \text{ MeV} \).

On the other hand, the decay of \( ^80\text{Br} \) to \( ^80\text{Se} \) can take place, both by \( \beta^- \) emission and electron capture:
\[ ^80\text{Br} \rightarrow ^80\text{Se} + \beta^- \]
\[ ^80\text{Br} + e^- \rightarrow ^80\text{Se} \]

The atomic mass difference between the two nuclei is 2.66 MeV which is greater than \( 2 m_e c^2 \).

5.6 Origin of continuous \( \beta \)-spectrum ; Neutrino hypothesis

As stated in § 5.3, the secondary electrons giving rise to the discrete peaks are not emitted from the \( \beta \)-disintegrating nuclei. Only the electrons in the continuous part of the \( \beta \)-spectrum (in Figs. 5.8 and 5.9) are emitted during \( \beta \)-disintegration of the nuclei.

The area under the energy distribution curves in Fig. 5.8 or 5.9 is proportional to the total number of electrons emitted. Usually the areas under the discrete peaks are small compared to the area under the continuous distribution graph (not more than a few percent), which shows that the number of secondary electrons forming the peaks is only a few percent of the total number of electrons emitted.

Careful measurements have shown that the total number of electrons, including those in the peaks as well as in the continuous spectrum, is slightly greater than the number of nuclei undergoing \( \beta \)-decay. The latter is found to be equal to the number of electrons in the continuous spectrum, which shows that the electrons emitted during \( \beta \)-decay form the continuous spectrum only, excluding the peaks.

Both for \( \beta^- \) and \( \beta^+ \) decays, the emitted \( \beta \)-particles are found to have continuous distribution of energies or velocities ranging from 0 up to a maximum. In the case of electron capture, no observable particle is emitted from the nucleus. Only x-ray photons or Auger electrons, characteristic of the product atom, are observed (see § 5.13).
We have seen before that during $\alpha$-integration, the mass number $A$ remains unchanged while the atomic number $Z$ changes by one unit. This means that either a neutron is changed into a proton (as in $\beta^-$ decay) or a proton is changed into a neutron (as in $\beta^+$ decay or in electron capture process) so that the total number of protons and neutrons $(Z + N = A)$ remains unchanged (see § 5.5).

Experimental study of $\alpha$-integration shows that the $\alpha$-particle spectra are discrete in nature, which points to the fact that the nuclei exist in discrete energy states, as expected from quantum mechanics for a closed micro-system. Transitions between these discrete levels in the parent and product nuclei give rise to the emission of mono-energetic groups of $\alpha$-particles (see § 4.11).

The same conclusion is drawn from the study of the $\gamma$-ray spectra (see Ch. VI). It may therefore be expected that due to the transition between such discrete energy levels in the parent and product nuclei in the $\beta$-decay process, the $\beta$-particles will also be emitted with one or more definite energies, given by the energy difference between the initial and final states, less the rest energy of the $\beta$-particle (see § 5.5). However, the observed continuous distribution of the $\beta$-energy is contrary to this expectation. Thus there is an apparent breakdown of the principle of conservation of energy in the case of $\beta$-decay (see Fig. 5.12). It may, however, be noted that the $\beta$-disintegration energy $Q_{\beta}$ defined in § 5.5 agrees with the mass-energy difference between the parent and the product nuclei less the electron rest-energy for both $\beta^-$ and $\beta^+$ disintegrations which also equals the maximum energy $E_m$ of the emitted $\beta$-particles.

Another puzzling feature about the $\beta$-decay is the apparent failure of the principle of conservation of angular momentum. We know that the protons and the neutrons, constituting the nucleus of an atom, have intrinsic spin angular momentum $s = \frac{1}{2}$ (in unit of $\hbar$) each. If the mass number $A$ of nucleons is even, then the total spin angular momentum of the nucleus $S = \Sigma s_j$ is either $0$ or integral. On the other hand, if $A$ is odd, then $S$ will be half odd integral. In addition, the nucleons may have other angular momentum $L$ which can only be integral multiples of $\hbar$. Since the total angular momentum of the nucleus (nuclear spin) is $I = L + S$, the value of $I$ is integral or half odd integral in units of $\hbar$. Depending on whether the number of nucleons $A$ in the nucleus is even or odd, respectively, For example, if $A$ is even, $I$ is integral or $0$. So in $\beta$-disintegration process, since $A$ remains unchanged, $I$ will remain integral or $0$, i.e., either $I$ does not change or changes by an integral multiple of $\hbar$. Similar is the case when $A$ is odd.

Now the electron has an intrinsic spin $\frac{1}{2}$. So during its emission from the nucleus, it can carry away a half odd integral unit of angular momentum, since the orbital angular momentum change, if any, can take place by an integral multiple of $\hbar$. This means that the emission of an electron from a nucleus should change the angular momentum by a half odd integral multiple of $\hbar$ which however contradicts the statement made above that in the $\beta$-disintegration process $I$ should change by an integral multiple of $\hbar$.

To explain these apparent inconsistencies, Wolfgang Pauli, in 1930 proposed that at the time $\beta^-$ decay of a nucleus, a hitherto unobserved second particle, in addition to the electron, is emitted, which carries away the balance of energy $E = E_m - E_n$ so that the total energy of the two particles is equal to the maximum $\beta$-energy $E_m$. When the electron is emitted with zero kinetic energy, the second particle is emitted with the maximum energy $E = E_m$. On the other hand, when the electron is emitted with the energy $E_n$, the other particle has energy $E = 0$.

Here we have neglected the energy of the recoil nucleus undergoing $\beta$-decay, since it is much heavier than the particles emitted (see below).

This new particle proposed by Pauli has been named the neutrino. It must have such physical properties that it would be very difficult to detect it. In fact it eluded observation for more than twenty five years after Pauli had proposed the neutrino hypothesis.

We can guess about some of the properties of the neutrino:

(i) The neutrino $(\nu)$ must be electrically neutral, so that the only change in the charge of the nucleus during $\beta$-decay is due to the emission of the electron or positron or due to the capture of an orbital electron. This is in agreement with observations.

(ii) The mass of the neutrino should be zero or very nearly so. This follows from the fact that the maximum energy $E_m$ of the emitted electron is equal to the mass energy difference between the parent and the product nuclei less the rest energy of the electron. If the neutrino had a finite mass then its rest energy must also be subtracted to get $E_m$ (see § 5.5).

(iii) Intrinsic spin of the neutrino should be $\frac{1}{2}$. Since the electron spin is also $\frac{1}{2}$, two spin $\frac{1}{2}$ particles are emitted during $\beta$-decay. Hence the two together will take away an integral unit of angular momentum, which is in agreement with the statement made above regarding the change of angular momentum in $\beta$-decay.

(iv) The neutrino must obey Fermi-Dirac statistics like the electron since its spin is $\frac{1}{2}$ (see § 2.9).
Enrico Fermi, the famous Italian physicist, was the first to work out a successful theory of \( \beta \)-decay, based on the neutrino hypothesis (1934). According to Fermi, \( \beta \)-decay occurs due to the transformation of a neutron into a proton inside the nucleus with the emission of an electron and an anti-neutrino (\( \bar{\nu} \)) which is the anti-particle of the neutrino, just as the positron is an anti-particle of the electron:

\[
n \rightarrow p + e^- + \bar{\nu} \quad \text{(5.6-1)}
\]

Such decay is actually observed in the case of a free neutron which has a half-life of 10.6 min, when it is outside the nucleus (see § 5.13). The reverse transformation of a proton into a neutron by the emission of positron and a neutrino also occurs inside the nucleus in \( \beta^+ \) decay:

\[
p \rightarrow n + e^+ + \nu \quad \text{(5.6-2)}
\]

However, this transformation cannot occur in the case of a free proton, because energy-conservation cannot be satisfied in this case, the proton being lighter than a neutron.

It may be noted that electrons, positrons, neutrinos and anti-neutrinos escaping from the nucleus in \( \beta \)-decay do not exist initially inside the nucleus. They are born at the time of \( \beta \)-decay, just as the photon is born at the time of radiative transition in an atom (or in a nucleus). This is unlike the case of \( \alpha \)-decay of a nucleus, since the two protons and two neutrons forming the \( \alpha \)-particle already exist inside the disintegrating nucleus.

Unlike \( \alpha \)-decay, which takes place only in the heaviest nuclei, \( \beta \)-decay can occur in a wide range of nuclei, starting from the lightest at \( A = 1 \) (in neutron) up to some of the heaviest nuclei known in nature or artificially produced.

The energy liberated in \( \beta \)-decay process also varies over very wide range. For example, in the decay \( ^{1}H \rightarrow ^{1}He + \beta^- \), the decay energy is only 0.02 MeV while in the decay \( ^{12}B \rightarrow ^{12}C + \beta^- \) it is 13.4 MeV.

A brief sketch of Fermi’s theory of allowed \( \beta \)-decay will be given in § 5.7. It gives a mathematical expression for the energy (or momentum), distribution of the \( \beta \)-particles.

The physical properties of the neutrinos are such that they are very difficult to detect. Since they carry no charge, they cannot produce ionization in matter. So the usual methods of detection of charged particles cannot be applied for their detection. Since they are practically massless, they cannot transfer energy to any other particle by elastic collision. Hence the method applicable in the case of detection of an electrically neutral particle like the neutron cannot be used in their case. There are reasons to believe that the sun emits a huge flux of neutrinos. For this reason, the earth is being incessantly bombarded by neutrinos. It has been estimated that about \( 10^{18} \) neutrinos pass through the human body every second. However, the probability of their interaction with the atoms in the human body is so small that not even one such collision takes place in a whole year. As stated before, there was no direct evidence for the detection of the neutrino for a long time. Finally in 1956, two American scientists, F. Reines and C.L. Cowan Jr., were successful in detecting the neutrino directly. Their experiment will be discussed in § 5.20.

As stated before, the neutrino (\( \nu \)) has an anti-particle, known as the anti-neutrino (\( \bar{\nu} \)). The former is emitted at the time of \( \beta^- \) decay and electron capture process, while the latter is emitted at the time of \( \beta^+ \) decay. We can represent these processes symbolically as follows:

\[
\beta^- \text{ decay:} \quad \left( X^+ \rightarrow Z^+ + Y^- + \beta^- + \bar{\nu} \quad \text{(5.6-3)}
\right.
\]

\[
\beta^+ \text{ decay:} \quad \left( X^- \rightarrow Z^- + Y^+ + \beta^+ + \nu \quad \text{(5.6-4)}
\right.
\]

\[
electron capture: \quad \left( X^+ + e^- \rightarrow Z^+ + Y^- + \nu \quad \text{(5.6-5)}
\right.
\]

The neutrino and anti-neutrino are both mass-less and charge-less particles with the same intrinsic spin (\( \frac{1}{2} \)). It is believed that the difference between them lies in the fact that the spin vector \( \vec{S} \) of the neutrino is antiparallel to its linear momentum \( p \), while for the anti-neutrino the two vectors are parallel, as shown in Fig. 5.13. See § 5.18.

Fig. 5.13. Neutrino and anti-neutrino.

We shall see later (Ch. XVIII) that the weakly interacting particles like the electron, positron, neutron, and antineutrino belong to a class of elementary particles called leptons. It is usual to associate a \( \text{lepton number} \ l \) with them. For the electron and the neutrino we put \( l = +1 \), for the antiparticles positron and antineutrino, we put \( l = -1 \). At the time of \( \beta \)-decay, there is conservation of the lepton number, which means that \( l \) remains the same before and after the decay. As an example, in \( \beta^- \) decay \( l = 0 \) on the left side of Eq. (5.6-3). On the right side, the total lepton number is \( 1 - 1 = 0 \). So the lepton number is conserved. Similarly for the \( \beta^+ \) decay. In the case of electron capture decay, \( l = +1 \) both on the left and right sides of Eq. (5.6-5). So lepton number conservation is satisfied.

5.7 Fermi’s theory of allowed \( \beta \)-decay

Enrico Fermi (1934) developed a theory for allowed \( \beta^- \) decay, based on Pauli’s neutrino hypothesis. Fermi used the result of Dirac’s time-dependent perturbation theory according to which the rate of transition from an initial state \( i \) to a final state \( f \) is given by (Golden Rule 2)§§

\[
w = \frac{2\pi}{\hbar} |H_{if}|^2 \rho(E) \quad \text{(5.7-1)}
\]

§§ See § 12.6 in Vol. I.
where $H_{ij}$ is the matrix element of the perturbing interaction $H$ causing the transition and is given by

$$H_{ij} = \int \psi_j^* H \psi_i \, d\tau$$  \hspace{1cm} (5.7-2)

$\rho(E)$ is the density of the final states. (See Quantum Mechanics § 12.6, Vol. I.)

Eq. (5.7-1) can be applied to the case of emission of electromagnetic radiation from an atomic (or nuclear) system. An excited atom in an initial state makes a radiative transition to the final state, which consists of the atom in the ground state plus the emitted photon.

According to Fermi, the $\beta$-emitting nucleus makes a transition such that the final state consists of the product nucleus plus the emitted electron and the neutrino. The initial state is of course the parent nucleus. The transition can be written as

$$X(A, Z) \rightarrow Y(A, Z \pm 1) + e + \nu$$  \hspace{1cm} (5.7-3)

where $e$ is either the electron ($\beta^-$) or the positron ($\beta^+$) while $\nu$ is either an antineutrino or a neutrino.

Since the emission of a particle is equivalent to the absorption of an antiparticle, we can represent the $\beta$-transformation process by the following equation in place of Eq. (5.7-3):

$$X(A, Z) + \nu \rightarrow Y(A, Z \pm 1) + e$$  \hspace{1cm} (5.7-4)

As we have seen in § 5.6, during $P^-$ transformation, a neutron in the nucleus is transformed into a proton with the creation of an electron-antineutrino ($\nu$) pair, while in $\beta^-$ transformation a proton in the nucleus is transformed into a neutron with the creation of a positron-neutrino pair. Using Eq. (5.7-4) we thus see that the initial state wave function $\psi_i$ and the leptonic wave function $\phi_e$ while the final state wave function $\phi_f$ should consist of the final state nuclear wave function $\psi_f$ and the leptonic wave function $\phi_p$.

In the case of radiative transition, the perturbation causing the transition is the electromagnetic field whose quantum manifestation is the photon. Its form is known. However in the case of $\beta$-emission, the nature of the perturbation is not known. It may be called the electron-neutrino field. Its quantum manifestation is the pair of these two particles.

The electron neutrino field interacts weakly with the nucleus, undergoing $\beta$-transformation and is known as the weak-interaction field. This follows from the fact the mean life of $\beta$-decay (minimum $10^{-3}$s) are long compared to the characteristic nuclear time, which is the time a nucleon to travel across the nucleus ($10^{-15}$).

$\beta$-decay theory is beset with another difficulty. The exact forms of the nuclear wave-functions are not known.

Fermi chose the simplest form of the electron-neutrino field $H$ which was relativistically invariant. The relativistic form of the theory must be chosen, because the electron and the neutrino both move at relativistic velocities. The matrix element can be written as

$$H_{ij} = \sum_n \int F(r_n) \psi_j^* U_n^* \psi_i \, d\tau$$  \hspace{1cm} (5.7-5)
expanding the expressions for \( \varphi_\beta \) and \( \varphi_\nu \) in Taylor series and neglect all terms other than the first:

\[
\varphi_\beta = \frac{1}{\sqrt{\Omega}} \left[ 1 + i k_\beta \cdot r_\beta + \frac{(i k_\beta \cdot r_\beta)^2}{2!} + \ldots \right]
\]

Similarly \( \varphi_\nu = \frac{1}{\sqrt{\Omega}} \) \( ...(5.7-9) \)

These approximations correspond to allowed transitions. Then the initial and final state wave-functions are

\[
\psi_i = u_i \varphi_\nu = u_i / \sqrt{\Omega} \quad \text{\ldots \ldots \ldots (5.7-11)}
\]

\[
\psi_f = u_f \varphi_\nu = u_f / \sqrt{\Omega} \quad \text{\ldots \ldots \ldots (5.7-12)}
\]

The matrix element then becomes

\[
H_{if} = \frac{8}{\Omega} \int u_f^* O u_i \cdot d\tau = \frac{8}{\Omega} M_{if} \quad \text{\ldots \ldots \ldots (5.7-13)}
\]

where

\[
M_{if} \text{ is the nuclear part of the matrix element excluding the strength factor } g \text{ and the normalization volume } \Omega.
\]

Calculation of the density of states:

The number of states of the neutrino in the momentum range \( p_\nu \) to \( (p_\nu + dp_\nu) \) when it is anywhere within the spatial volume \( \Omega \) is given by (see Ch. XX, Vol. I)

\[
dN_\nu = 4\pi p_\nu^2 dp_\nu \Omega / h^3 \quad \text{\ldots \ldots \ldots (5.7-15)}
\]

Similarly for the electrons, the number of states in the momentum interval \( p_\beta \) to \( (p_\beta + dp_\beta) \) is

\[
dN_\beta = 4\pi p_\beta^2 dp_\beta \Omega / h^3 \quad \text{\ldots \ldots \ldots (5.7-16)}
\]

Thus the total number of states available for the \( \beta \) and \( \nu \) in the above momentum ranges, when they are confined within the volume \( \Omega \) is

\[
dN = dN_\beta dN_\nu = \frac{16\pi^2 \Omega^2}{h^6} p_\beta^2 p_\nu^2 dp_\beta dp_\nu \quad \text{\ldots \ldots \ldots (5.7-17)}
\]

To get the density of states, we have to divide \( dN \) by the energy interval \( dE_\beta \) for the electrons:

\[
\rho(E) = \frac{dN}{dE_\beta} \quad \text{\ldots \ldots \ldots (5.7-18)}
\]

But since \( E_\beta + E_\nu = E_m \) where \( E_m \) is a constant

\[
dE_\beta + dE_\nu = 0
\]

Beta Particles and Beta Activity

Since \( E_\nu = c p_\nu = E_m - E_\beta \), we have

\[
dE_\beta = -dE_\nu = -c d p_\nu
\]

So we get

\[
\rho(E) = \frac{16\pi^2 \Omega^2}{(2\pi \hbar)^6} \frac{p_\beta^2 p_\nu^2 dp_\beta dp_\nu}{c d p_\nu} \quad \text{\ldots \ldots \ldots (5.7-19)}
\]

Substituting for \( p_\nu \) we get finally

\[
\rho(E) = \frac{\Omega^2}{4 \pi^4 \hbar^6 c^3} p_\beta^2 (E_m - E_\beta)^2 dp_\beta \quad \text{\ldots \ldots \ldots (5.7-20)}
\]

So by using Eqs. (5.7-14) and (5.7-20) we get from Eq. (5.7-1) the probability of emission per second of the \( \beta \)-particles in the momentum range \( p_\beta \) to \( (p_\beta + dp_\beta) \) as

\[
P(p_\beta) dp_\beta = \frac{2\pi \beta^2}{\hbar^2} \frac{1}{\Omega^2} \frac{p_\beta^2}{4 \pi^4 \hbar^6 c^3} \frac{p_\beta^2 (E_m - E_\beta)^2 dp_\beta}{c \hbar^2} \quad \text{\ldots \ldots \ldots (5.7-21)}
\]

This expression is independent of \( \Omega \) as it should be.

Eq. (5.7-21) gives the Fermi distribution for the emitted \( \beta \)-particles in allowed \( \beta \)-decay.

Coulomb corrections:

We have seen that the electronic wave function \( \varphi_\beta \) is affected by the Coulomb field of the nucleus and hence cannot be a plane wave as assumed above. Actually there are two types of Coulomb corrections:

(i) \( \varphi_\beta \) is affected by the nuclear electric field. The simplest assumption would be to treat the electron as subject to the central electrostatic field of a point nucleus and ignore the electric quadruple and magnetic moments. The radial wave-functions in this case differ from the plane wave functions in two ways. Firstly they have different normalization near the origin since, due to the Coulomb acceleration, the time spent by them near the nucleus is different. Secondly the relativistic electrons have different radial dependence.

The Coulomb correction can be estimated in the non-relativistic limit as follows:

\[
F(Z, E) = \frac{1}{|\varphi_\beta(0)|^2} \frac{1}{|\varphi_\beta(0)|^2} \frac{1}{|\varphi_\beta(0)|^2} = \frac{2\pi \eta}{1 - \exp(-2\pi \eta)} \quad \text{\ldots \ldots \ldots (5.7-22)}
\]

where \( |\varphi_\beta(0)|^2 \) is the electron wave function in the absence of the Coulomb effect. Here \( \eta = Z \varepsilon^2 / h\nu \) for electrons and \( \eta = -Z \varepsilon^2 / h\nu \) for positrons. \( Z \) is the atomic number of the product nucleus, \( \varepsilon \) is the velocity of the electrons far away from the nucleus. The Coulomb correction increases the probability of electron emission and decreases the
When the relativistic effect is taken into account, the nuclear Coulomb correction is ~ 5%. It does not depend on \( \varepsilon_p \) appreciably, except at very low energy ( < 35 keV). For low energy positron emission, the correction factor may be as large as 10 for high Z.

(ii) Another effect requiring correction is the screening of the nuclear charge by the atomic electrons, which is more important for positron emission than for electron emission. For electron emission, the maximum correction is ~ 5%. It does not depend on \( \varepsilon_p \) appreciably, except at very low energy ( < 35 keV). For low energy positron emission, the correction factor may be as large as 10 for high Z.

If the Coulomb correction is taken into account, the r. h. s. of Eq. (5.7-21) should be multiplied by the function \( f(Z, E_p, \varepsilon_p) \) or \( F(Z, p_p) \) so that we get

\[
P(p_p)dp_p = \frac{g^2}{2 \pi^3 c^3 \hbar^3} |M_{eff}|^2 \frac{1}{1 + \frac{2p_p}{m_e c}} F(Z, p_p) p_p^2 (E_m - E_p)^2 dp_p \quad \text{(5.7-24)}
\]

If the number of \( \beta^- \)-particles emitted per second in the momentum range \( p_p \) to \( (p_p + dp_p) \) is \( N(p_p) dp_p \) and the total number emitted per second is \( N_0 \), then

\[
N(p_p) dp_p = N_0 P(p_p) dp_p = A F(Z, p_p) p_p^2 (E_m - E_p)^2 dE_p \quad \text{(5.7-25)}
\]

where \( A \) is a constant given by

\[
A = \frac{g^2 N_0}{2 \pi^3 c^3 \hbar^3} |M_{eff}|^2 \quad \text{(5.7-26)}
\]

The value of \( |M_{eff}|^2 \) is uncertain in the absence of any knowledge regarding the form of the interaction operator \( Q \) or the nuclear wave functions \( u_i \) and \( u_{ip} \). For allowed transition \( |M_{eff}|^2 = 1 \) which makes

\[
A = \frac{g^2 N_0}{2 \pi^3 c^3 \hbar^3} \quad \text{(5.7-27)}
\]
As $E_p \to 0$, this makes $N(E_p) \to 0$ because of the exponential factor.

Fig. 5.14. (a) Fermi energy-distribution of the $\beta$-particles. The effects of the Coulomb force on the emitted $\beta^+$ and $\beta^-$ are shown. The dashed curve is obtained by neglecting Coulomb effect. (b) Momentum distribution.

These features of the $\beta$ - distribution curves are shown in Fig. 5.14a. It may be noted that instead of $E_p$, if we plot the momentum $p_\beta$ along the abscissa, then the numbers of both $\beta^+$ and $\beta^-$ are zero as $p_\beta \to 0$, even when the effect of the Coulomb force is taken into account (see Fig 5.14 b).

Kurie plot:

Eq. (5.7-25) gives

$$\left[ \frac{N(p_\beta)}{p_\beta F(z, p_\beta)} \right]^{1/2} = K \left( E_m - E_p \right)$$

...(5.7-36)

Here $K$ is a constant. Eq. (5.7-36) shows that a plot of the function on the left hand side of the equation against $E_p$ is a straight line, with a negative slope, which intersects the abscissa at $E_p = E_m$. This type of graph is known as a Kurie plot. In Fig. 5.15 is shown the Kurie plot for allowed $\beta$-decay, which is a straight line, in agreement with the theory.

Sometimes, deviations are observed in straight line Kurie plots. This may be due to the following two reasons:

(i) A forbidden transition will generally result in a concave-up type graph. The simple theory for allowed transition is to be modified in this case.

(ii) Another reason for the deviation is a complex $\beta$-spectrum due to transitions to two or more states of the daughter nucleus (see Fig. 5.16 a). The situation is analogous to the emission of $\alpha$-rays with a fine structure. The transitions to the upper states are usually followed by the emission of $\gamma$-rays or internal conversion electrons. (see Fig. 5.16 b).

Neutrino mass:

It may be noted that the theoretical shape of the $\beta$-distribution curve near the end-point $E_p = E_m$ depends on the neutrino mass $m_\nu$.

A simple calculation shows that in this case, the density of states $\rho(E_m)$ is proportional to the neutrino momentum, $i.e.$, to $(E_m - E_p)^{1/2}$. The beta distribution curve, for $m_\nu \neq 0$ then approaches the upper limit with a vertical tangent, instead of a horizontal tangent for $m_\nu = 0$. This is shown schematically in Fig. 5.17. The most recent experiments of Lubinov and other (1980) on the $\beta$ decay $^3\text{H} \rightarrow ^3\text{He}$ gives a value 14 eV $< m(\bar{\nu}) < 46$ eV.

Fig. 5.17. Effect of finite neutrino mass on $\beta$-distribution curve.

(a) For $m_\nu = 0$ (b) For $m_\nu \neq 0$.

5.8 Allowed and forbidden $\beta$-transitions

Fermi's theory of $\beta$-decay gives a relationship between the decay constant $\lambda$ of the nucleus and the maximum $\beta$-energy $E_m$. Integrating Eq.
(5.7-24), we get the total probability per second of the emission of \( \beta \)-particles of all momenta from 0 up to the maximum \( p_m \):

\[
\lambda = \int_0^p P(p_B) \, dp_B
\]

\[
P_m = C \int_0^p F(Z, p_B) \rho_B^2 (E_m - E_p)^2 \, dp_B
\]

where

\[
C = \frac{\hbar^2}{2 \pi^2 c \hbar^2} |M_{if}|^2
\]

The integration is most conveniently carried out by using the following transformations:

\[
e_p = E_p / m_e c, \quad \eta_p = p_p / m_e c
\]

\( e_p \) and \( \eta_p \) are dimensionless forms for the \( \beta \)-energy and momentum. Using Eq. (5.3-3) we get

\[
E_m - E_p = W_m - W_p = \sqrt{p_m^2 c^2 + m_e^2 c^4} - \sqrt{p_p^2 c^2 + m_e^2 c^4}
\]

\[
= \frac{m_e c^2}{\sqrt{1 + \eta_m^2} - \sqrt{1 + \eta_p^2}}
\]

where the subscript \( m \) refers to the maximum \( \beta \)-energy and momentum. Then

\[
\lambda = \frac{\hbar^2}{2 \pi^2 c \hbar^2} |M_{if}|^2 \int_0^p \frac{F(Z, \eta)}{\eta^2} m_e c^2 \eta^2 d\eta
\]

\[
= \frac{\hbar^2 c^4 m_e^2}{2 \pi \hbar^2} |M_{if}|^2 \int_0^\eta (\sqrt{1 + \eta_m^2} - \sqrt{1 + \eta^2})^2 \eta^2 d\eta
\]

For low \( Z \) elements, \( F(Z, \eta) = 1 \). We then get

\[
\lambda = \frac{\hbar^2 c^4 m_e^2}{2 \pi \hbar^2} |M_{if}|^2 \int_0^\eta (\sqrt{1 + \eta_m^2} - \sqrt{1 + \eta^2})^2 \eta^2 d\eta
\]

\[
= \frac{\hbar^2 c^4 m_e^2}{2 \pi \hbar^2} |M_{if}|^2 f(\eta_m)
\]

Thus the probability of \( \beta \)-decay increases with increasing value of \( |M_{if}|^2 \). The factor multiplying \( f(\eta_m) \) on the r.h.s. of Eq. (5.8-5) is a constant if \( |M_{if}|^2 \) is assumed to be a constant. Since the total probability of the disintegration per second is nothing but the decay constant \( \lambda = 0.693 / \tau \), \( \tau \) being the half-life, we get

\[
\frac{\hbar^2 c^4 m_e^2}{2 \pi \hbar^2} |M_{if}|^2 f(\eta_m) = \ln \frac{2}{\tau}
\]

or

\[
f(\eta_m) \tau = \frac{2 \pi \hbar^2 \ln 2}{\hbar^2 c^4 m_e^2 |M_{if}|^2} = \text{constant}
\]

\[
(5.8-6)
\]

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The value of the constant on the r.h.s. of Eq. (5.8-6) depends on the square of the matrix element \( |M_{if}|^2 \), which in turn depends on the wavefunctions of the parent and product nuclei, as also on the strength of the weak interaction causing \( \beta \)-decay.

The product \( f(\eta_m) \tau \) is known as the comparative half-life. Eq. (5.8-6) shows that for larger value of \( |M_{if}|^2 \) the comparative half-life is smaller. So small value of \( f(\eta_m) \tau \) indicates larger probability of \( \beta \)-decay (larger \( \lambda \)) and vice-versa. The comparative half-life can be estimated by evaluating the integral in Eq. (5.8-5):

\[
f(\eta_m) = \int_0^\eta \left( \frac{1}{\sqrt{1 + \eta_m^2}} - \frac{1}{\sqrt{1 + \eta^2}} \right)^2 \eta^2 d\eta
\]

\[
= -\frac{1}{4} \eta_m - \frac{1}{12} \eta_m^3 + \frac{1}{30} \eta_m^5
\]

\[
+ \frac{1}{4} \sqrt{1 + \eta_m^2} \ln (\eta_m + \sqrt{1 + \eta_m^2})
\]

...(5.8-7)

Eq. (5.8-7) gives the approximate limiting values \( f(\eta_m) \) for large and small \( \eta_m \):

For \( \eta_m >> 5 \), \( f(\eta_m) = \frac{1}{30} \eta_m^5 \)

For \( \eta_m << 0.5 \), \( f(\eta_m) = \frac{2}{\eta_m^2} \)

...(5.8-8)

Between the limiting values, \( f(\eta_m) \) has to be calculated by using the more correct formula (Eq. 5.8-7). We give below in Table 5.1 the values of \( f(\eta_m) \) for different \( \eta_m \).

<table>
<thead>
<tr>
<th>( \eta_m )</th>
<th>0.05</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(\eta_m) )</td>
<td>1.48 x 10^-11</td>
<td>1.9 x 10^-8</td>
<td>2.44 x 10^-7</td>
<td>4.17 x 10^-6</td>
<td>3.12 x 10^-5</td>
<td>1.4 x 10^-3</td>
</tr>
<tr>
<td>( \eta_m )</td>
<td>1.0</td>
<td>2.0</td>
<td>3.0</td>
<td>4.0</td>
<td>5.0</td>
<td>6.0</td>
</tr>
<tr>
<td>( f(\eta_m) )</td>
<td>1.153 x 10^-2</td>
<td>0.707</td>
<td>6.54</td>
<td>29.96</td>
<td>95.45</td>
<td>243.5</td>
</tr>
</tbody>
</table>

In the calculation of \( f(\eta_m) \) above, we have assumed a plane wave for the electron. If however, the distortion due to the Coulomb field is taken into account, the factor \( F(Z, \eta) \) must be retained in the integrand of Eq. (5.8-4). The integration then gives a function \( f(Z, \eta_m) \) in place of \( f(\eta_m) \) so that we get

\[
f(Z, \eta_m) \tau = \frac{2 \pi \hbar^2 \ln 2}{\hbar^2 c^4 m_e^2 |M_{if}|^2} = \text{constant}
\]

...(5.8-9)

The constant on the r.h.s. involves \( |M_{if}|^2 \) which, as we have seen, cannot be evaluated exactly. For allowed transitions this is usually taken to be unity.
Using the value of the integral given by Eq. (5.8-7) we can estimate the comparative half-lives for different nuclides from a knowledge of their half-lives. This helps us determine the constant $g$ giving the strength of the weak interaction (see end of § 5.9).

If the values of $(\log_{10} \tau g)$ are calculated for different $\beta$-emitters, using the experimental values of $p_n$ and $\tau$, then the values are found to be grouped in certain regions, as shown in Fig. 5.18. On the basis of such grouping, we can broadly classify the $\beta$-emitters into different categories as follows:

(i) $\log \tau g = 3$ to 4: The $\beta$-transitions are most probable for the nuclides in this group. They are called allowed and favoured or super-allowed. The number of nuclides in this group is relatively small.

(ii) $\log \tau g = 4.5$ to 5: The $\beta$-transitions are allowed but not favoured or simply allowed. Their number is the largest. The probability of transition for these nuclides is lower than for (i).

(iii) $\log \tau g = 7$ to 9: These constitute the first forbidden transitions, the transition probability being lower than that for (i) or (ii).

For still higher values of $\log \tau g$ we get forbidden transitions of higher order, for which the transition probabilities are still lower.

The above classification of the transitions is not always very clear-cut and rigid, as will be evident from Fig. 5.18.

In Table 5.2 are included some $\beta$-emitters, listed according to their $\log \tau g$ values.

The classification of the $\beta$-transitions as allowed or forbidden is similar to that in the case of radiative transitions in the atoms, being governed by the value of the matrix element $M_{hf}$ for the transitions. As we have seen, $M_{hf}$ includes the time-dependent perturbing potential, which depends on the electron and neutrino wave-functions. Assuming plane waveforms for $\Psi_p$ and $\Psi_\nu$, we can expand them as in Eqs. (5.7-9) and (5.7-10). In the previous discussions we retained only the first term in the expansion which is just unity, corresponding to allowed transition. In this case, the emitted particles ($\beta$ and $\nu$) do not carry any orbital angular momentum, so that $\Delta L = 0$. The successive terms in the expansion fall off by a factor $kR = R/\lambda$, where $R$ is the nuclear radius and $\lambda = 2 \pi \alpha$ is the de Broglie wavelength of the emitted particles. For the $\beta$-disintegration, the energies of the order of 1 MeV, $R/\lambda \sim 0.1$, so that the successive terms in the expansion decrease by a factor of about $1/10$.

The probability for the allowed transition depends on the nuclear matrix element $M_{hf}$ which is given by Eq. (5.7-14). If this happens to be zero, then we have to consider the higher order terms in the expansions of $\Psi_p$ and $\Psi_\nu$. As we saw, the next higher order term is lower by a factor $R/\lambda \sim 1/10$ so that the probability of this first forbidden transition, is lower by a factor of $(R/\lambda)^2 \sim 1/100$. Though this is small compared to the probability of the allowed transition it is not as small as in the case of the radiative transition, in the atoms for which $(R/\lambda)^2 = (10^{-8}/10^{-5})^2 = 10^{-6}$. So the forbidden transitions are more important for $\beta$-decay.

If the second term in the expansion also gives a value zero for $M_{hf}$ then we have to take the third term corresponding to the second forbidden transition with a probability lower again by a factor $\sim 1/100$.

For successive higher order terms in the expansion, we get the third, fourth etc. forbidden transitions, the probability in each case being lower by a factor $\sim 1/100$ than in the previous order.

The orbital angular momenta carried away by the emitted particles are 1, 2, 3, etc. units of for the first, second, third etc. forbidden transitions respectively.

In evaluating the matrix element of the transitions, another correction has to be taken into account, which was not considered above. This is a relativistic correction term, which is of the order $(v/c)$ and which should be present in the matrix element, $\nu$ being the velocity of the nucleons. Since the square of the matrix element is involved in the probability of transition and since $v/c \sim 1/10$, this correction results in terms again $\sim 1/100$ which is as large as the other correction term.

5.9 Selection rules in $\beta$-decay

The $\beta$-transitions are governed by certain selection rules, depending on whether the $\beta$-decay is allowed or forbidden.

A nuclear state is characterized by the total angular momentum $I$ and the parity $\Pi$. The total angular momentum or the "spin" of the nucleus is the vector sum of the orbital and spin angular momenta: $I = L + S$. When all the transition takes place from an initial state $I_i$ to a final state $I_f$ we can write the change in $I$ as $\Delta I = I_f - I_i$ such that

$$\Delta I = \Delta L + \Delta S$$

...(5.9-1)
As we have seen, the electron and the neutrino wavefunctions $\psi_\beta$ and $\psi_\nu$ can be represented by plane waves, if the Coulomb effect is neglected for $\psi_\beta$. In the expansion of the plane wave, the first term gives the allowed transition, which corresponds to the emission of $\beta$ and $\nu$ in the S-state with $l = 0$. This can be seen by writing:

$$\psi = \exp(i k \cdot r) = \exp \left( i \frac{p \cdot r}{\hbar} \right)$$

$$= 1 + \frac{i}{\hbar} (p \cdot r) + \frac{i^2}{\hbar^2} (p \cdot r)^2 + \cdots \quad \text{(5.9-2)}$$

Now $pr$ is a measure of the orbital angular momentum which in quantum theory can be written as $il$, $l$ being the orbital angular momentum quantum number of the emitted $\beta$ and $\nu$. Neglecting all but the first term in the expansion, then amounts to taking only $l = 0$ term, since $r = 0$ for this.

The higher order terms correspond to higher values of $l$. The orbital angular momentum carried away by the electron-neutrino pair must be equal to the change $\Delta L$ of the nucleus in $\beta^-$ transition. So, for allowed transition $\Delta L = l = 0$. Then from Eq. (5.9-1), we get for allowed transition $\Delta I = \Delta S$. In Fig. 5.19 we show the vector diagram for the angular momentum change in $\beta^-$ transition $\Delta I = \Delta S$. The corresponding vector diagrams should be as shown in Fig. 5.19b. The electron and neutrino are spin $1/2$ particles. Their spins can be aligned either parallel or anti-parallel to each other, so that the total spin angular momentum carried away by them (see Fig. 5.19c and d) can be either $S = 0$ (parallel) or $S = 1$ (anti-parallel). In the first case, the spin change of the nuclear state will be $\Delta I = \Delta S = 0$. This corresponds to what is known as the Fermi selection rule. In the second case $\Delta I = |\Delta S| = 1$ (in unit of $\hbar$) which corresponds to Gamow-Teller selection rule. The possible changes in the nuclear spin in the case of G-T selection rule are shown by the vector diagram in Fig. 5.20. From the figure it is clear that the change in $l$ can have the values $\Delta I = 0, \pm 1$ in this case.

As a result, a change in the nuclear spin in the $\beta^-$ transition $\Delta S = 0$ or 1 can be denoted by $\Delta I = 0, \pm 1$. In the case of Fermi selection rule, the transition cannot take place from the state $I_f = 0$ to the state $I_i = 0$ because this transition involves the spin angular momentum change of one unit ($|\Delta S| = 1$).
As stated before, the nuclear states are also characterized by the parity $\Pi$ ($l^\Pi$). For states with even (odd) $L$, the parity is even (odd). Since no change of $L$ takes place in the case of allowed transitions ($L_i = L_f; \Delta L = 0$), there is no change of parity. This is expressed by the word "No" in the above selection rules.

To illustrate the above selection rules, we take the case of the $\beta$-decay of $^6$He:

\[ ^6\text{He} \rightarrow ^6\text{Li} \]

\[ (l_f = 0^+) \quad (l_f = 1^+) \]

There is a spin change $\Delta l = 1$ without any change of parity (no) in this transition. So this transition is governed by pure allowed Gamow-Teller selection rule. That it is an allowed transition is confirmed by the $\tau$ value given in Table 5.2 which shows it to be superallowed. An example of pure Fermi selection rule is observed in the decay of $^{14}$O:

\[ ^{14}\text{O} \rightarrow ^{14}\text{N}^+ \] (excited state).

Here the transition is from the ground state $l^I = 0^+$ of $^{14}$O to an excited state of $^{14}$N with $l^I = 0^+$. So the spin change is $\Delta l = 0$ without any change of parity (no). Hence this transition is governed by pure allowed Fermi selection rule ($\tau$ value in Table 5.2 confirms that it is a super-allowed transition).

That G-T selection rule is not applicable in this case is evident from the fact that it is the case of $l_i = 0$ to $l_f = 0$ transition.

It may be noted that there is also a pure G-T allowed transition to the ground state ($1^+$) of $^{14}$N in this case for which $\Delta l = 1$, no.

In some allowed transitions, both Fermi and G-T selection rules (mixed selection rules) are applicable. Some examples are given below:

\[ ^{1}n \rightarrow ^{1}\text{H} + \beta^- (1/2^+ \rightarrow 1/2^+) \]

\[ ^{1}\text{H} \rightarrow ^{3}\text{He} + \beta^- (1/2^+ \rightarrow 1/2^+) \]

\[ ^{35}\text{Cl} \rightarrow ^{35}\text{S} + \beta^- (3/2^+ \rightarrow 3/2^+) \]

Each of the above transitions is permitted by both Fermi and G-T selection rules. $\tau$ values in Table 5.2 show that the first two are super-allowed, while the last is an allowed but not favoured transition.

Selection rules applicable in the cases of forbidden transitions are given below.

First forbidden transitions: In this case the orbital angular momentum changes by 1 unit, since the $\beta$ and $\nu$ take away one unit of orbital angular momentum. So we have the following selection rule:

(a) Fermi selection rule: Since $\Delta S = 0, \Delta I = 0, \pm 1$; yes; $0 \rightarrow 0$ is forbidden.

(b) G-T selection rule: Since $\Delta S = 1$, we have $\Delta I = 0, \pm 1 \pm 2$, yes.

Some examples of first forbidden transition are listed in Table 5.2. The selection rules are summarized in Table 5.3.

### Table 5.3

<table>
<thead>
<tr>
<th>Transition</th>
<th>Fermi selection rules</th>
<th>Gamow-Teller selection rules</th>
</tr>
</thead>
<tbody>
<tr>
<td>Allowed</td>
<td>$\Delta l = 0$, parity change $\Delta l = 0$</td>
<td>$\Delta l = 0$, parity change $\Delta l = 0$</td>
</tr>
<tr>
<td>First forbidden</td>
<td>$\Delta l = 0$, parity change $\Delta l = 0$</td>
<td>$\Delta l = 0$, parity change $\Delta l = 0$</td>
</tr>
</tbody>
</table>

In these mixed transition cases, the nuclear matrix element can be written as

\[ M_H = C_F^2 M_F^2 + C_{GT}^2 M_{GT}^2 \]

where $C_F$ and $C_{GT}$ are coupling constants in units of $g$ and

\[ M_{GT} = \frac{B}{x} \]

\[ \tau = \frac{1}{x} \]

The constant $B$ is given by

\[ B = \frac{2 \pi^3 \hbar^2 \ln 2}{g^2 c^4 m_e^2 (C_F^2 + C_{GT}^2)} \]

and

\[ x = \frac{C_{GT}^2}{C_F^2 + C_{GT}^2} \]

The constants $B$ and $x$ can be obtained from the experimental results for pure Fermi ($^{14}$O $\rightarrow ^{14}$N) and pure G-T ($^6$He $\rightarrow ^6$Li) $\beta$-decays and the mixed transitions as in the $^1n \rightarrow ^1H$ or $^1H \rightarrow ^3$He decay. The values of the nuclear matrix elements in these cases are known because of the simple nuclear structures. The results give $x = 0.56$ and

\[ B = B_F = 1.403 \times 10^{-62} \text{ J} \cdot \text{m}^3 \]

The value of $g_{GT}$ is slightly greater. Two values have been deduced from two sets of experiments on neutron half-life:

\[ g_{GT}/g_F = 1.244 \] (Chistensen)

\[ g_{GT}/g_F = 1.279 \] (Spirak)

5.10 Sargent diagrams

It may be noted that classification of $\beta$-transitions on the basis of the relationship between the half-lives of the $\beta$-emitters and the end-point
β-energies \( (E_m) \) was first proposed by B.W. Sargent (1933). The empirical Sargent rules, similar to the Geiger-Nuttall relationship in α-decay, are usually depicted in the form of graphs between \((\log \lambda)\) and \((\log E_m)\) which are known as Sargent diagrams and have the forms shown in Fig. 5.21. As can be seen from the diagram, the points corresponding to the different β-emitters fall on (or near) two different lines. These lines do not correspond to the different natural radioactive series, unlike in the case of the α-emitters. Rather they correspond to half-lives (or decay constants) differing by a factor of about 100 for the same \( E_m \). The transitions in the case of the points on the upper graph (shorter half-lives) correspond to the allowed β-transitions while those on the lower graph (longer half-lives) correspond to forbidden transitions.

![Sargent diagram](image)

Fig. 5.21. Sargent diagram.

As we have seen such classification of the β-emitters according to the degree of forbiddenness of the β-transitions is predicted by the theory of β-decay, which thus provides a theoretical explanation of Sargent's rules.

5.11 Nature of the nuclear matrix element

The value of the nuclear matrix element \( M_{hf} \) depends on the extent to which the initial nuclear wave function overlaps the final wave function. In general, since the nuclear wave functions are not known, not much can be said about the extent of the overlap from theoretical considerations. We can get some idea about it only from the experimental value of the β-decay rates. However, in the case of mirror nuclei, the wave functions of the initial and final states are very similar, as can be understood from the following simple considerations.

![Wave functions](image)

Fig. 5.24. Wave functions of the transformed nucleus in the parent and product nuclei, when there is very little overlapping.

Assuming the protons and neutrons to fill up the successive space states of the respective potential wells from bottom upwards, as shown in Fig. 5.22, each state can be occupied by only two nucleons of the same kind with opposite spins \( \frac{1}{2} \) because of Pauli's exclusion principle. Then in a nucleus with widely different numbers of protons \( (Z) \) and neutrons \( (N) \), the numbers of proton levels and neutron levels which are filled up are considerably different, as shown in Fig. 5.23 for the case \( Z=6 \) and \( N=12 \). As seen in the figure the three lowest proton levels and the six lowest neutron levels are completely filled up in this nucleus. When it undergoes β-transformation, a neutron in the uppermost filled level transforms into a proton, as shown in the middle part of the figure. However, since the two lower proton levels are empty, this proton from the uppermost level subsequently goes down to the fourth level, so that the final state is as shown on the right hand side of the figure for \( Z=7 \) and \( N=11 \). Obviously the initial and final states are quite different in this transformation, so that there is little overlap between the wave functions of the two (see Fig. 5.24).

On the other hand, for \( Z-N=1 \) as in a mirror nucleus, the same numbers of proton and neutron levels are completely filled from bottom upwards with one extra proton in the uppermost level, as can
be seen from Fig. 5.25a for the case $Z=8, N=7$. When this nucleus undergoes $\beta^+$-transformation, the eighth proton in the uppermost level is transformed into a neutron, so that the final state has $Z=7$ and $N=8$. The neutron produced by the transformation of the proton then occupies the uppermost fourth level, as shown on the r.h.s. of the figure (b). The proton and neutron distributions for the parent and product nuclei are then very similar. The only difference is that in place of a proton in the uppermost fourth level of the former we have a neutron in the uppermost fourth level of the latter (see Figs. 5.25a and b).

![Diagram](image)

**Fig. 5.25.** (a) and (b) Proton neutron levels in the $\beta^+$-decay of a mirror nucleus ($^1\text{O} \rightarrow ^1\text{N}$). (c) Wavefunctions of the transformed nucleon in the parent and product nuclei, when there is considerable overlapping.

Hence the wave functions for the two nuclei must be very similar, if we neglect the Coulomb effect, so that there is considerable overlap of the two (Fig. 5.25 c). Thus the value of $|M_d|^2 = \int \psi^*_d \psi^* \, d\tau$ is the highest in this case, making it a super-allowed transition.

### 5.12 Orbital electron capture

We have seen that the electron capture type of $\beta^+$-decay takes place when the positron emission is not energetically possible. When a nucleus captures an orbital electron, only a neutrino is emitted, which is extremely difficult to detect. So the only way of detecting an electron capture decay is to observe the characteristic X-rays of the daughter atom. When the electron is captured from the K-shell, characteristic K X-rays are emitted, as the vacancy in the K-shell is filled up by the electrons from the outer L, M etc. shells. The energies of the X-ray photons will be $B_K - B_L$, $B_K - B_M$, etc., where $B_K$, $B_L$, $B_M$ etc. are the electron binding energies in the corresponding shells. There is also an alternative process to the X-ray emission. The excess energy of the atom following the K-capture may be directly transferred to an L-electron, which is thereby emitted. This is known as an Auger electron and the process as Auger or radiationless transition (see Vol I).

**Beta Particles and Beta Activity**

Electron capture probability depends on the two factors: (a) the probability of the electron being at the position of the nucleus and (b) the probability of capture of the electron by the nucleus. Since an electron in the K shell has the largest probability of being at the position of the nucleus, so in most cases, it is K-capture which is observed. However, the nucleus may also capture an L-electron, though with much lower probability, because the L-electron is much farther away from the nucleus than the K-electron. L-capture is followed by the emission of characteristic L X-rays of the daughter atom.

The emission of the characteristic X-rays or of the Auger electrons following K-capture merely indicates that such capture has taken place. It does not give any idea about the energy-change taking place. This can be found by measuring the energy of the product nucleus recoiling in a direction opposite to the direction of emission of the neutrino. This has actually been done by (a) applying of retarding potential to stop the recoil nucleus and (b) by measuring the time of flight of the recoil nucleus from the moment of X-ray emission.

The probability of K-capture depends on the nature of the electron-cloud around the nucleus (see Vol I). Thus the same nuclear transformation occurs with different probabilities (i.e. different half-lives if the nature of the electron-cloud is changed due to molecular binding of the radioactive atom with other atoms. This effect was actually observed by E. Segre, C. E. Wiegand and R. F. Leininger (1951) in the K-capture of the $^7\text{Be}$ ($Z=4$) nucleus: $^7\text{Be} + e_K \rightarrow ^7\text{Li} + \nu$. The effect is expected to be appreciable for light elements. A difference of about 0.08% has been found between the decay rates of $^7\text{Be}$ in BeF$_2$ and in beryllium metal.

### 5.13 Theory of orbital electron capture

The probability of orbital electron capture by a nucleus can be calculated by using the result of the time-dependent perturbation theory, as was done in the case of $\beta^+$-decay:

$$P = \lambda = \frac{2\pi}{\hbar} |H_{\beta}|^2 \rho(E)$$

We shall develop below the theory of K-capture. The theory is relatively simpler than in the case of $\beta$-emission, since only one particle (neutrino) is emitted in this case. Since it is a two body process, the neutrino is emitted with a definite energy $E_{\nu}$ given by

$$E_{\nu} = E_0 + m_e c^2 - B_K$$  \hspace{1cm} (5.13-1)

where $E_0$ is the nuclear transformation energy, equal to the mass energy difference between the parent and product nuclei. $B_K$ is the binding energy of the electron in the K-shell, which is captured by the nucleus.

Since in the final state a neutrino of definite-energy is emitted, the density of states $\rho (E)$ corresponds to that for the neutrino of momentum
The number of neutrino states in the momentum range \( p \) to \( (p + dp) \) is

\[
dN_\nu = \frac{4 \pi p^2 dp \times \Omega}{(2 \pi \hbar)^3} = \frac{4 \pi p^2 dp \times \Omega}{(2 \pi \hbar)^3} \quad \text{...}(5.13-2)
\]

Here we assume the whole transformation to take place within the spatial volume \( \Omega \). Transforming to energy we get

\[
dN_\nu = \frac{4 \pi E_\nu^2 dE_\nu \times \Omega}{(2 \pi \hbar)^3} \quad \text{...}(5.13-3)
\]

The density of states is then

\[
\rho(E_\nu) = \frac{dN_\nu}{dE_\nu} = \frac{E_\nu^2 \Omega}{2 \pi \hbar^3} \quad \text{...}(5.13-4)
\]

As in the case of \( \beta \)-emission, the transformation of the parent nucleus into the product takes place by weak interaction. See Eq. (5.7-6). This contains the electron and neutrino wave functions \( \phi_e \) and \( \phi_\nu \), which in the "allowed approximation" have to be evaluated at the position of the transforming nucleon (proton). As before we take a plane wave for the neutrino which gives for \( r = \sqrt{\Omega} \),

\[
\phi_\nu(r) = \frac{1}{\sqrt{\Omega}} \exp(ik_\nu r) = \frac{1}{\sqrt{\Omega}}
\]

For the electron, we take the K-shell wavefunction for a hydrogen-like atom (see Vol.I):

\[
\phi_e = \phi_K = \frac{1}{\sqrt{\hbar}} \left( \frac{Z}{a_0} \right)^{3/2} \exp(-Zr/a_0) \quad \text{...}(5.13-5)
\]

where \( a_0 = 4 \pi e^2 h^2/\mu_e c^2 \) is the Bohr radius. \( Z \) is the atomic number of the parent nucleus. For \( r = 0 \), this becomes

\[
\phi_e(0) = \frac{1}{\sqrt{\hbar}} \left( \frac{m_e Z e^2}{4 \pi e^2 h^2} \right)^{3/2} \quad \text{...}(5.13-6)
\]

The probability of K-electron capture per second is then given by

\[
\lambda_K = \frac{2 \pi e^2}{\hbar} | \phi_e(0) |^2 | \phi_\nu(0) |^2 | M_{if} |^2 \rho(E_\nu) \\
= \frac{Z^2 g^* m_e^2 e^4 E_\nu^2}{32 \pi^2 e_0^2 c^2 h^5} | M_{if} |^2 \quad \text{...}(5.13-7)
\]

Hence we have taken into account the fact that there are two electrons in the K-shell.

If the energy \( E_\nu \) is expressed in units of \( m_e c^2 \) we get

\[
\lambda_K = \frac{g^* c Z^2 m_e^2 e^4 (E_\nu + 1 - B_K)^2}{32 \pi^2 e_0^2 c^2 h^5} | M_{if} |^2 \quad \text{...}(5.13-8)
\]

The above theory can be adapted to the case of the capture of an electron from orbits of larger principal quantum numbers (e.g., L-capture) for allowed transition, without much difficulty in the non-relativistic approximation. (see \textit{Physics of the Nucleus} by M. A. Preston).

Writing

\[
f_K = \frac{Z^2 e^4 (E_\nu + 1 - B_K)^2}{16 \pi^2 e_0^2 c^3 h^3} = 4 \pi (\alpha Z)^2 (E_\nu + 1 - B_K)^2 \quad \text{...}(5.13-9)
\]

where \( \alpha \sim 1/137 \) is Sommerfeld's fine structure constant, we can write

\[
\frac{E_\nu}{\Omega} = \frac{\lambda K}{\tau_K} = CF_K \quad \text{...}(5.13-10)
\]

\( \tau_K \) is the half-life of K-capture and \( C^* \) is a constant. We then have

\[
f_K \cdot \tau_K = \frac{\Omega}{C^*} = \text{constant} \quad \text{...}(5.13-11)
\]

where \( C = 2 \pi^2 h^3 \ln 2 / g^* m_e^2 \), \( | M_{if} |^2 \), \( m_e c^2 \) is the binding energy of the K-electron is (in the unit of \( m_e c^2 \))

\[
B_K = \frac{1}{m_e c^2} \frac{m_e^2 e^4}{32 \pi^2 e_0^2 c^2 h^3} = \frac{Z^2 e^4}{32 \pi^2 e_0^2 c^2 h^3} \left( \frac{(\alpha Z)^2}{2} \right) \quad \text{...}(5.13-13)
\]

The numerical values of the product \( f \tau \) determine whether the K-capture transitions are allowed or forbidden (see § 5.8). For example, in the transformation \( ^7\text{Be} + e_K \rightarrow ^7\text{Li} \), it is found that the transformation energy \( E_0 = 0.3508 \text{ MeV} = 0.686 m_e c^2 \) while \( \tau_K = 53.28 \text{ d} \). Since the K-shell binding energy in beryllium (\( Z = 4 \)) is only 112 eV, this can be neglected so that we get

\[
f_K = 4 \pi (\alpha Z)^2 (E_\nu + 1)^2 \\
= 4 \pi \left( \frac{4}{137} \right)^2 \times (1.686)^2 = 8.9 \times 10^{-4}
\]

\[
\tau_K = 53.28 \text{ d} = 4.603 \times 10^6 \text{ s}
\]

Hence

\[
f_K \tau_K = 4097
\]

and

\[
\log f_K \tau_K = 3.61
\]

This shows it to be a super-allowed transition. Since it is a case of \((3/2)^-\) to \((3/2)^-\) transition, it is a mixed Fermi and G. T. super-allowed transition. The value of \( f_K \tau_K \) given shows that the strength of the interaction causing the orbital electron capture must be of the same order of magnitude as in the case of \( \beta \)-particle emission.
It may be noted that the expression for $\lambda_k$ involves the square of the nuclear matrix element $|M_{nl}|^2$ which is the same that appears in the expression (Eq. 5.8-5) for the $\beta$-decay probability $\lambda_{\beta}$. In the ratio $\lambda_k/\lambda_{\beta}$, this cancels out. So the relative probabilities of K-capture and $\beta^+$ decay for the same nucleus (K to $\beta^+$ branching ratio) is independent of $|M_{nl}|^2$ which, as we have seen, cannot be determined accurately. We get

$$\frac{\lambda_k}{\lambda_{\beta}} = \frac{Z^2 e^6 (E_0 + 1 - B_k)}{16 \pi^2 m_e^2 c^2 n^2 c^7 f (Z, \eta_m)} \quad \text{(5.13-14)}$$

For light elements and high $\beta$-end points, K-capture is less probable. On the other hand, the K-capture process becomes relatively more important than $\beta^+$ decay for high $Z$ elements, specially when the nuclear transformation energy approaches $\beta^+$ threshold. Further, Eq. (5.13-8) shows that the K-capture process is in general more important for high $Z$ elements.

### 5.14: Absorption of Electrons in Matter

#### Ionization loss:

The mechanism of energy-loss by ionization is basically the same for high energy electrons, as in the case of the heavier charged particles (see § 4.12). However there are two factors which modify the stopping power formula (4.12-23). Because of the small electron mass, relativistic variation of the mass with energy has to be considered at relatively much lower energies (of the order of an MeV). Also because the collision in this case is between two identical particles, quantum mechanical exchange effect must be taken into account. This was done by Moller (1932), on the basis of which the following formula was derived by Bethe (1933)

$$\frac{dE}{dx}_{\text{ion}} = \frac{1}{(4 \pi \varepsilon_0)^2} \cdot \frac{2 \pi e^4 N Z}{m_e v^2} \left[ \ln \frac{2 \varepsilon v^2}{2Z(1-\beta^2)} - (2\sqrt{1-\beta^2} - 1 + \beta^2) \ln 2 + 1 - \beta^2 + \frac{1}{8} (1 - \sqrt{1-\beta^2})^2 \right] \quad \text{(5.14-1)}$$

Here $\beta = v/c$ and $E$ is the relativistic kinetic energy of the electron given by

$$E = \frac{m_e c^2}{\sqrt{1 - \beta^2}} - m_e c^2$$

For the non-relativistic case, the above equation reduces to Eq. (4.20-22). For ultra-relativistic case ($\beta = 1$), we get

$$\frac{dE}{dx}_{\text{ion}} = \frac{1}{(4 \pi \varepsilon_0)^2} \cdot \frac{2 \pi e^4 N Z}{m_e v^2} \left[ \ln \frac{E^2}{2Z(1-\beta^2)} + \frac{1}{8} \right] \quad \text{(5.14-2)}$$

### Beta Particles and Beta Activity

**Radiation loss:** Apart from the ionization-loss discussed above, a relativistic electron loses energy by another process known as the radiation loss.

According to the electromagnetic theory, a light charged particle, like the electron, when subjected to an acceleration or deceleration, loses energy by the emission of electromagnetic radiation. This process is known as bremsstrahlung and the resulting loss of energy as radiation loss.

The amount of energy radiated per second by a particle of charge $q$ moving with an acceleration $f$ is given by

$$W = \frac{q^2 f^2}{6 \pi \varepsilon_0 c} \quad \text{(5.14-3)}$$

The above expression, which is classical holds for a non-relativistic particle. If $F$ be the force, we have $f = F/m_0$ where $m_0$ is the rest mass of the particle.

This means that the energy radiated in a Coulomb collision of the particle with another charged particle is inversely proportional to the square of the mass of the particle and directly on the square of its charge. Thus the radiation-loss is important for a light charged particle, like the electron. Further, it is of importance in collisions with the atomic nuclei only and not with the atomic electrons. For a nuclear charge $Ze$, the radiation-loss in a collision with the nucleus is greater than that due to collision with an electron by a factor $Z^2$, because the force $F$ depends on $Ze$.

The above considerations also apply if a particle moves with relativistic velocity and when quantum mechanical considerations are taken into account. Radiation-loss becomes particularly important at higher energies ($E >> 2m_e c^2$). There is a critical energy $E_{cr}$ at which the rate of radiation-loss becomes equal to the rate of ionization-loss. Hence for $E >> E_{cr}$ the radiation-loss predominates. For electrons, $E_{cr} \approx 10$ MeV for lead the air $E_{cr} \approx 100$ Mev. It is found that the specific energy-loss due to radiation is given by (see Ch. XIX)

$$\frac{dE}{dx}_{\text{rad}} \propto EZ^2 \quad \text{(5.14-4)}$$

In the region where $E >> E_{cr}$, the energy of the particle decreases exponentially:

$$E = E_0 \exp(-x/L_r) \quad \text{(5.14-5)}$$

Here $L_r$ is the thickness of the absorber required to reduce the energy of the particle to 1/eth of its initial value, and is known as the radiation-length. The value of $L_r$ depends on the substance. For air $L_r = 300$ m; for lead $L_r = 0.5$ cm.

* See Classical Electricity and Magnetism by W. Panofsky and M. Phillips.
The total rate of energy loss by an electron is given by
\[ \frac{dE}{dx} = \frac{dE}{dx}_{\text{ion}} + \frac{dE}{dx}_{\text{rad}} \] (5.14-6)

The two types of energy losses are shown graphically in Fig.5.26 for electrons in lead.

**Fig. 5.26. Different types of energy losses by electrons in matter.**

**Range of electrons**: Electrons do not have very well-defined range. They lose a much larger fraction of their energy in each collision than the heavier charged particles. This produces a much larger straggling in their range. Further, they suffer frequent large angle scattering, specially at low energies, when the ionization-loss predominates. Because of this, the actual path-lengths traversed by the electrons in a substance are much greater than the thickness of the material which brings them to the end of their path.

This is illustrated in Fig.5.27. It is thus not possible to calculate theoretically the range of the electrons as a function of their energy, unlike in the case of the heavier charged particles (see § 4.12). Fig. 5.27. Actual path of an electron in matter.

This is usually done empirically on the basis of the experimental data.

As we have seen the measured range of the electrons in a substance is usually much smaller than the actual path traversed by them which may be from 1.5 to 4 times the former. Thus range measurements of the electrons, except in cloud-chamber photographs, do not have the same definiteness as in the case of the heavy charged particles. Even so, careful measurements on electrons of definite initial energies (as in the cases of conversion electrons) have been used to define the maximum extrapolated range \( R_e \) for the electrons.

In spite of the variations in the form of the absorption curves, the thickness of the material required to reduce the number of electrons traversing the material to zero is usually found to have a definite value. This is the maximum extrapolated range referred to above and is usually a linear function of the initial energy with different parameters for the straight line (slope and intercepts) in different energy regions.

It is more important to know the range in the case of the \( \beta \)-particles from a radioactive substance, which have a continuous distribution of energies. The shape of the absorption curve in this case is markedly different from that of the homogeneous electrons. The absorption curves in both the cases are shown in Fig. 5.28 for comparison. Since the lower energy electrons are more readily absorbed, the absorption curve for the continuous distribution falls off more rapidly than for the homogeneous electrons (see Fig. 5.28b).

**Fig. 5.28.** (a) Absorption curve of a homogeneous group of electrons in matter. (b) Absorption curve of \( \beta \)-rays in matter. Notice the exponential decrease of the intensity.

The absorption curves for the continuous \( \beta \)-rays are nearly exponential in appearance over the major portion of its length so that one can define an absorption coefficient \( \mu \) for a particular spectrum and represent the intensity by the formula
\[ I = I_0 e^{-\mu x} \]

If the absorber thickness \( x \) is measured in metre, \( \mu \) has the dimension of the reciprocal of length \( (\text{m}^{-1}) \).

The mass absorption coefficient defined by the relation \( \mu_m = \mu / \rho \) where \( \rho \) is the density of the substance, can be represented by a relation of the type
\[ \mu_m = k/E_m^{1.33} \] (5.14-7)

where \( k \) is a constant and \( E_m \) is the energy in MeV. \( \mu_m \) has the dimension of metre\(^2\) per kg.
Because the shape of the absorption curve depends somewhat on the geometry of the measuring arrangement, a more reproducible quantity is the maximum range \( R \) which is the maximum thickness of the material required to stop the \( \beta \)-rays of the highest energy \( E_m \).

N. Feather found a linear relationship between \( R \) and the energy \( E_m \). Later, the following formula due to Glendenin and Coryell was found to give a better fit:

\[
R = 5.42 E_m - 1.33 \quad \text{(5.14-8)}
\]

Here \( E_m \) is in MeV and \( R \) is in kg/m\(^2\). The above relation is valid in the energy range 1 to 3 MeV. For \( E_m < 0.8 \) MeV, the following formula represents the range better:

\[
R = 4.07 E_m^{1.38} \quad \text{(5.14-9)}
\]

It may be noted that when expressed in kg/m\(^2\), the range is almost the same in all materials.

The ranges of the \( \beta \)-particles are much longer than those of the \( \alpha \)-particles of comparable energies. For example, the range of the \( \beta \)-particles emitted by the artificially radioactive isotope \(^{32}\)P \((Z = 15)\) having the maximum energy 1.71 MeV is about 810 mg/cm\(^2\) in aluminium. The reason for the \( \beta \)-particle ranges being much longer than those of the \( \alpha \)-particles is that their ionizing-power is much smaller. It is only about 1/200 of that of the \( \alpha \)-particles. \( \beta \)-particles cannot produce ions below a critical energy. Above this critical energy, their ionizing power increases up to a maximum and then begins to decrease and again increases slowly. The maximum ionizing power is attained at the energy of about 1000 eV. At this energy the ionizing power (specific ionization) is about 100 ions per mm in air at S.T.P. At higher energies, this is only about 5 ions per mm in air at S.T.P.

### 5.15 Half-life of the free neutron

We have stated before that the free neutron is not stable. It decays by \( \beta \) emission into a proton according to the following scheme:

\[
\beta^- \rightarrow \nu + p^+ + \gamma
\]

The half-life of the neutron was first determined by J. M. Robson, a Canadian scientist, in 1951 by using the high neutron flux from a nuclear reactor. The thermal neutron flux from the reactor was \( 1.5 \times 10^{14} \) neutrons per m\(^2\) per second. The neutrons were allowed to pass through a vacuum chamber within which they produced the protons by \( \beta \)-disintegration (see Fig. 5.29). The protons had the maximum kinetic energy of about 1000 eV. These were accelerated through a potential difference of 13 kilo-volts and then deflected by a magnetic field at \( P \) to be focussed into an electron multiplier \( D_1 \) which detected them. The \( \beta \)-particles produced in the disintegration of the neutrons were similarly focused on an anthracine scintillator \( D_2 \) by a magnetic field in the magnetic spectrometer \( B \).

The pulses due to the \( \beta \)-particles were detected in coincidence with the proton pulses with a delay to allow for the time of flight of the slow moving protons. Even with the high flux of neutrons, the chance of the neutron decaying within the small volume viewed by the detectors was so small that only a few coincidences per hour were detected.

In Fig. 5.30 the Kurie plot obtained by Robson in the \( \beta \)-disintegration of the neutron is shown.

The half-life determined by Robson was 12.8 ± 2.5 min. The value was later improved by measurements at Oak Ridge in the U.S.A. and in Russia. The currently accepted value is

\[
\tau = 10.13 \pm 0.09 \ \text{min}
\]

### 5.16 Parity non-conservation in \( \beta \)-decay

The concept of parity of a wave-function was discussed in § 11.18 of Vol.I. It has been seen that if the parity operator commutes with the Hamiltonian of a system, then parity is conserved and \( P \) is a good quantum number. There are strong experimental evidences to show that parity conservation holds in electromagnetic and strong interactions. Hence it was believed that parity conservation was a universal law and therefore held for weak interaction also, causing radioactive \( \beta \)-decay and weak decay of the elementary particles. However, about 1955 an unexpected development took place which threw doubts about the validity of this assumption.

Fermi's theory of \( \beta \)-decay and its further modifications were based on the tacit assumption of the conservation of parity in weak interaction. The experimental facts, as were available in those days, could be explained on the basis of these theories.

In the early nineteen fifties, two mesons were discovered in the cosmic rays, which were called \( \tau \) and \( \theta \) mesons. They had almost the same mass, same half-life and same spin, but had different decay modes as shown below:

\[
\tau^+ \rightarrow \pi^+ + \pi^0
\]

\[
\theta^+ \rightarrow \pi^+ + \pi^0 + \pi^0
\]

The pions, which are produced in the above decays, have spin-parity 0. These are weak decays and involve zero orbital angular momentum change.
Since the parity quantum numbers are multiplicative, it follows that $x^*$ must have spin-parity $0^-_\pi$ and $0^-_\pi$ should have spin parity $0^-$. So it would be tempting to conclude that these were two different particles. However, the identical values of their mass (493.667 MeV), half-life ($1.237 \times 10^{-8}$ s) and spin (0) are strong reasons to believe that they are one and the same particle.

In later years they have been named K-mesons ($K^*$). See Ch. XVIII.

T.D. Lee and C.N. Yang, two American scientists of Chinese descent, first proposed that these were the same particles, but the different modes of their decay indicated that parity was not conserved in these decays which involved weak interaction.

The above facts are known as the $\tau$-0 puzzle. Since $\beta$-decay in radioactivity also involved weak interaction, it was natural to search for some evidence of non-conservation of parity in weak interaction in the experimental results on $\beta$-decay. Lee and Yang searched, for such evidence, but could find none. So they suggested an experiment involving $\beta$-decay which would verify whether their hypothesis was correct or not (1957). The experiment was subsequently performed at the National Bureau of Standards in Washington in the U.S.A. by C.S. Wu and her associates (1957), which gave a brilliant confirmation of Lee and Yang's suggestion. The experiment is described below.

It may be mentioned that the discovery of non-conservation of parity in weak interaction constitutes one of the most fundamental discoveries in the second half of the present century. Lee and Yang were awarded Nobel prize in physics in 1957 for this very important discovery.

**Experiment of Wu and others**

The experimental arrangement used by C.S. Wu, E. Ambler, R.W. Hayward, D.D. Hopp and R.P. Hudson, is shown in Fig. 5.31. They used a $^{60}$Co source as a $\beta$-emitter, which has a half-life of 5.3 yr emitting $\beta$-rays of maximum energy $0.31$ MeV. The decay scheme $^{60}$Co $\rightarrow$ $^{60}$Ni $+ \beta^-$ is shown in Fig. 5.32. The spin change $\Delta I = 1$. It is an allowed transition ($\Delta L = 0$) of the pure G-T type.

The most important requirements of the experiment were the production of extremely low temperature (0.01 K) and very high magnetic field ($-10$ T) for the polarization (i.e. preferential alignment of the spins) of the $^{60}$Co nuclei.

They used the method of adiabatic demagnetization of a paramagnetic material (cerium magnesium nitrate) for the production of the very low temperature.

The production of the very high magnetic field involved the use of paramagnetic materials the atomic electrons within which were polarized to produce magnetic fields of the order 10 T in the vicinity of the nuclei. The magnetic moments of these electrons could easily be polarized by the application of a weak magnetic field ($-10^{-2}$ T) at the temperature of 0.01 K.

The $^{60}$Co source (S) in the form of a thin film deposited on a single crystal of Ce-Mg-nitrate was housed within a glass vacuum chamber. An anthracine crystal (C) of thickness 1.6 mm was placed inside the chamber at a distance of 2 cm to detect the $\beta$-particles. The light from the scintillations produced in it by the $\beta$-particles passed through a long lucite rod to fall on the cathode of a photo-multiplier tube placed outside the chamber. The chamber was at first cooled to liquid helium temperature.

Ce-Mg nitrate was magnetized by a special magnet. Subsequently cooling by adiabatic demagnetization brought down the temperature to the desired value after the helium gas filling the chamber had been pumped out. The magnet was then removed and a magnetic field produced by a solenoid was used to polarize the electrons which provided...
the magnetic field for polarizing the $^{60}$Co nuclei. The direction of polarization of the nuclei could be reversed by reversing the magnetic field.

The intensity of the $\beta$-particles was measured both in the forward and backward directions w.r.t. the direction of nuclear polarization when the temperature was very low (~0.01 K). Subsequently the sample was allowed to warm up by introducing helium gas into the chamber and the intensities in the two directions were measured as functions of the warming up time which was about 8 min.

The results of the experiment are shown in Fig. 5.33a, in which the $\beta$-intensity $I_{\beta}$ is plotted as a function of warming up time $t$ both for forward and backward directions. The results show that the $\beta$-particles are emitted preferentially in the backward direction w.r.t. to the direction of nuclear polarization, indicated in the figure by the direction of the polarizing magnetic field $H$. The anisotropy in the $\beta$-intensity for the cold sample disappears as the sample warms up.

The degree of polarization of the source nuclei was measured from the anisotropy of the $\gamma$-ray intensities emitted from $^{60}$Co by means of the two NaI(Tl) scintillation counters shown in Fig. 5.31, parallel (B) and perpendicular (A), to the direction of polarization. The results are shown in Fig. 5.33b which show anisotropy in the $\gamma$-intensity as recorded by the counters, both for the cold as well as for the warm samples. The anisotropy disappears when the sample warms up.

**Interpretation of Wu's experiment:**

Parity operation leads to inversion at the origin of the coordinate system, i.e. the vector $r$ is changed to $-r$ which is equivalent to changing the spherical polar coordinates $(r, \theta, \phi)$ to $(r, \pi-\theta, \pi+\phi)$.

If there is cylindrical symmetry, then there is no $\phi$ dependence of a given wave function so that there is only $r$ and $\theta$ dependence. The fact that in the experiment of Wu et al there is a symmetry in the intensity distribution in the forward and backward directions w.r.t. the direction of polarization of the $^{60}$Co nuclei implies that the intensity can be expressed by the formula

$$I_{\beta} = I_{\beta}^0 (1 + acos \theta)$$

where $a < 0$. This gives greater intensity at $\pi - \theta$ than at $\theta$.

That the experiment of Wu et al really shows parity non-conservation can be understood by referring to Fig. 5.34 which shows the schematic picture of $\beta$-decay of $^{60}$Co nucleus ($5^+$) to the exited state $4^+$ of $^{60}$Ni. The spin change $\Delta I = 1$ is taken up by the electron and anti-neutrino emitted with parallel spins as shown in the Fig. 5.34 since $\Delta L = 0$.

Now the linear momentum $p = dr/dt$ is a polar vector and hence it changes sign on reflection. But the angular momentum $L = r \times p$ is an axial vector and hence does not change sign on reflection. This is also true for the spin angular momentum. So in the mirror image of the $\beta$-decay $^{60}$Co $\rightarrow ^{60}$Ni $+ \beta^- + \bar{\nu}$, the nuclear spins of the parent and product nuclei (shown by double arrows) do not change directions. Similarly the spins $S_p$ and $S_v$ of the $\beta^-$ and the antineutrino $\bar{\nu}$ do not change direction in the mirror image. However their linear momenta $p_p$ and $p_\bar{\nu}$ shown by single arrows, change sign on reflection. Considering $\beta^-$ emission in the backward direction, $p_p$ and $p_\bar{\nu}$ have opposite directions, as shown in the figure, $(p_\bar{\nu}$ must be parallel to $S_v$) so that momentum may be conserved with $p_p$ opposite to the spin of the residual nucleus $^{60}$Ni. Then due to reasons mentioned above, $p_p$ must have the same direction as the $^{60}$Ni spin in the mirror image (i.e. aligned parallel).
This is exactly what happens on reversing the polarizing field in Wu's experiment, which depicts a situation corresponding to the mirror image of what happened before the reversal.

If parity is conserved in a given interaction, then the results of an experiment performed in the laboratory should be identical with those observed in the mirror image situation. Since the experiment of Wu et al. showed that there was asymmetry in β-intensity between the two directions, the result of the experiment proved conclusively that parity is not conserved in β-decay which is caused by weak interaction.

In the case of β⁺ decay, the positrons are emitted preferentially in the forward direction w.r.t. the direction of nuclear polarization. This has been shown by using 58Co source (τ = 72 d) which decays as follows:

\[ {_{58}^{58}\text{Co}} \rightarrow {_{58}^{58}\text{Ni}} + {\beta^+} + \nu. \]

### 5.17 Helicity of the neutrino

Fig. 5.34 shows the relative orientation of the spins and linear momenta of the electron and the antineutrino during β⁻ decay. Both are longitudinally polarized which means that their spins are aligned parallel (for ν⁻) or antiparallel (for β⁻) to their respective linear momenta for backward emission of the β⁻ particle.

Though the polarization of the β-particles observed in β-decay is not a general characteristic of the electrons, it is regarded as a basic property of the antineutrino (and of the neutrino) and is known as helicity (H).

We define helicity as

\[ H = \frac{\sigma \cdot p}{|\sigma| |p|} \]  

...(5.17-1)

where \( \sigma \) is the spin of the neutrino and \( p \) is its momentum. The Pakistani physicist Abdus Salam and L. Landau of Russia were the first to point out, on the basis of the two component theory of the neutrino* (assuming the neutrino mass \( m_\nu = 0 \)), that helicity should be a fundamental property of the neutrino. To understand this we note that for a particle with a finite mass, the velocity can be different in different frames of reference. If, for instance, the particle spin is parallel to its momentum in some frame of reference, then to an observer moving faster than it in the previous frame, it will appear to be moving in the opposite direction and hence has momentum opposite to its spin. So it will have different polarizations in the different frames of reference. However, this is not the case for a massless particle (fermion) which must always move with the velocity of light \( c \). Hence there can be no frame of reference which will move faster than it.

From the expression given above, the helicity is \( H = \pm 1 \) depending on whether the relative orientation of the spin and momentum of the particle is parallel or antiparallel. According to the two component theory mentioned above, the neutrino has \( H = -1 \) (antiparallel orientation of spin relative to momentum) while the antineutrino has \( H = +1 \) (parallel orientation of spin relative to momentum). This is the only distinction between the two particles.

If spin is regarded as a rotation, then the motion of the neutrino is analogous to that of a left handed screw while the motion of the antineutrino is similar to the motion of a right handed screw.

The existence of a definite helicity of the neutrino is directly related to the violation of parity conservation in weak interaction. If a particle has right-left symmetry, then upon mirror reflection, the wave-function either remains the same or simply changes sign, while the particle is transformed to itself. However, a particle with a definite helicity does not possess right-left symmetry. So upon mirror reflection, a right handed screw-like particle transforms into a left handed screw-like particle. Thus the particle is not transformed to itself, which means violation of parity conservation.

### Measurement of neutrino helicity:

The helicity of neutrino was measured directly in an experiment performed by M. Goldhaber, L. Grodzins and A.W. Sunyar (1958). They used as source the K-capturing \( {^{152}\text{Eu}} \) (\( \tau = 9.3 \) h) isomer which has the decay scheme shown in Fig.5.35.

The product nucleus \( {^{152}\text{Sm}} \) goes to the ground state by γ-emission (\( E_\gamma = 961 \) keV) which is an E1 transition. In the experiment, resonance fluorescence produced by the γ-rays were studied (see § 6.16). Because of the recoil of the emitting \( {^{152}\text{Sm}} \), the energy was reduced by about \( E_r = 3.26 \) eV from the transition energy as can be calculated with the help of Eq. (6.16-1). Because of the very short half-life of the excited state, its width \( \Gamma \) is relatively large, being about 0.02 eV. Even so, it is not wide enough to compensate for the recoil energy-loss (\( \Gamma^2 < E_\gamma \)). However, the compensation is provided by Doppler shift due to the recoil velocity of the source, which is a product in the K-capture decay of the parent nucleus \( {^{152}\text{Eu}} \) emitting a neutrino of energy \( E_\nu = 900 \) keV (method 'c' of compensation discussed in § 6.16).

This recoil energy is about 2.86 eV. Notice that there are two different types of recoil of the \( {^{152}\text{Sm}} \) nucleus due to two different reasons: the first due to neutrino emission in the electron capture by the parent nucleus and the second due to γ-emission from the excited product nucleus \( {^{152}\text{Sm}} \).

Since the compensation due to Doppler shift is slightly less than the recoil energy change of the γ-rays, the γ-rays were allowed to proceed at
an angle slightly less than 180° w.r.t. the direction of emission of the neutrino in K-capture decay of $^{152}$Eu, as can be seen from Fig. 5.36a showing the experimental arrangement of Goldhaber et al. The theme of the experiment is illustrated in Fig. 5.37.

We can write down the law of conservation of angular momentum in the two successive transitions involved as below:

$^{152}$Eu$^m + e_K \rightarrow ^{152}$Sm$^* + \nu$

In this case angular momentum will be conserved if the spins of the neutrino and the $^{152}$Sm$^*$ nucleus are oriented oppositely. Since their momenta are also in opposite directions, it follows that the longitudinal polarization of the nucleus $^{152}$Sm$^*$ must have the same sign as that of helicity of the neutrino. In the second transition we have

$^{152}$Sm$^* \rightarrow ^{152}$Sm + $\gamma$

So the experiment boils down to the measurement of the sign of the circular polarization of the $\gamma$-rays. This is measured from the change in the number of counts in the $\gamma$-ray detector upon a reorientation of the magnetic field in the magnetized iron. The results are shown in Fig. 5.36b.

The helicity of the neutrino was found to be negative. Though there is no direct experimental determination of the helicity of the anti-neutrino, all other experimental data show that it must be positive.

It was seen in § 5.6 that each lepton has a lepton number (also called the leptonic charge). For electrons and $\nu_e$ the leptonic charge is +1 while for positrons and $\bar{\nu}_e$ it is -1. We conclude that the sign of longitudinal polarization of the electronic leptons is opposite to the sign of their leptonic charge.

5.18 Nature of the neutrino : Double beta decay

Lepton number conservation mentioned in § 5.6 requires us to postulate that the neutrino and antineutrino are different. The so-called
double beta decay process provides a confirmation of this hypothesis and is therefore a check on the form of the beta decay theory that must be accepted.

Consider the three nuclei $^{130}\text{Te}$, $^{130}\text{Xe}$, and $^{136}\text{Xe}$. The middle one is an odd-odd nucleus which has a higher mass than the other two $e-e$ nuclei. Further it is known that of the two $e-e$ nuclei $^{130}\text{Te}$ and $^{136}\text{Xe}$, the former has a larger mass than the latter. So it is conceivable that $^{130}\text{Te}$ may directly transform to $^{130}\text{Xe}$ by double $\beta$ decay process. Written in terms of the fundamental $\beta$-decay process (Eq. 5.6-1) we can therefore write the double $\beta$-decay by the equations

\[
n_1 \rightarrow p_1 + \beta^-_1 + \bar{\nu}_1
\]

\[
n_2 \rightarrow p_2 + \beta^-_2 + \bar{\nu}_2
\]

and

\[
^{130}\text{Te} \rightarrow ^{130}\text{Xe} + 2\beta^- + 2\bar{\nu}
\]  
...(5.18-1)

Thus four leptons (two $\beta^-$ and two $\bar{\nu}$) are emitted.

The above equation holds if, according to our previous discussion, the neutrino ($\nu$) and antineutrino ($\bar{\nu}$) are different from one another. Now the $\beta^-$ decay can be considered either as being accompanied by an antineutrino emission or by a neutrino absorption. The alternative process can be written as

\[
n + \nu \rightarrow p + \beta^-
\]  
...(5.18-2)

If the neutrino and antineutrino are identical ($\nu = \bar{\nu}$) then the process written above (Eq. 5.18-2) would be

\[
n \rightarrow p + \beta^- + \bar{\nu} \equiv p + \beta^- + \nu\n\]  
...(5.18-3)

The final result of the double $\beta$-decay will then be the transformation of two neutrons in the parent nucleus into two protons in the product nucleus with the emission of two $\beta^-$ particles but without the emission of any neutrino (or antineutrino) :

\[
^{130}\text{Te} \rightarrow ^{130}\text{Xe} + 2\beta^-
\]  
...(5.18-4)

Thus only two particles ($2\beta^-$) are emitted which carry away the whole energy of double $\beta$-transformation.

The two possible processes discussed above have however markedly different probabilities of occurrence. If $\nu$ and $\bar{\nu}$ are different, then the probability of double $\beta$ decay will be much lower than if $\nu = \bar{\nu}$. This is due to the fact that the virtual neutrino (no emission of neutrino) has a much larger region of phase space accessible to it than an emitted neutrino. The half-life $\tau$ of double $\beta$-decay according to the virtual neutrino theory would be around $10^{16}$ to $10^{17}$ y while $\tau$ should be about $10^{22}$ to $10^{23}$ y if $\nu$ and $\bar{\nu}$ are different. The experimental results seem to be in favour of the latter hypothesis, as the following table shows.

### Table 5.4

<table>
<thead>
<tr>
<th>Double $\beta$ transition</th>
<th>$\tau$ (theor) for $\nu = \bar{\nu}$ (y)</th>
<th>$\tau$ (theor) for $\nu \neq \bar{\nu}$ (y)</th>
<th>$\tau$ (exp) (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^{130}\text{Te} \rightarrow ^{130}\text{Xe}$</td>
<td>$2 \times 10^{16}$</td>
<td>$4 \times 10^{21}$</td>
<td>$2.5 \times 10^{21}$</td>
</tr>
<tr>
<td>$^{82}\text{Se} \rightarrow ^{82}\text{Kr}$</td>
<td>$10^{16}$</td>
<td>$10^{22}$</td>
<td>$6 \times 10^{19}$</td>
</tr>
</tbody>
</table>

### 5.19 Experiment for neutrino detection

The basic $\beta$-transformation equations in nuclei can be written as (see Eq. 5.6-1) and (5.6-2)

\[
n \rightarrow p + \beta^- + \bar{\nu}
\]

and

\[
p \rightarrow n + \beta^+ + \nu
\]
or equivalently as

\[
n + \nu \rightarrow p + \beta^-
\]  
...(5.19-1)

\[
p + \bar{\nu} \rightarrow n + \beta^+
\]  
...(5.19-2)

In all the above four cases, the lepton number conservation is satisfied.

If $\nu$ and $\bar{\nu}$ are identical ($\nu = \bar{\nu}$), then in the latter two equations, we could write $\bar{\nu}$ in place of $\nu$ and $\nu$ in place of $\bar{\nu}$ on the left hand sides. However, as seen from double $\beta$-decay process discussed, the following two processes should be forbidden :

\[
n + \nu \rightarrow p + \beta^-
\]  
...(5.19-3)

\[
p + \nu \rightarrow n + \beta^+
\]  
...(5.19-4)

Processes (5.19-1) and (5.19-2) may be regarded as inverse $\beta$ transformations.

Since the cross-section for the interaction of neutrino (or antineutrino) with nuclei is very small, being about $10^{-46} - 10^{-45} \text{ m}^2$, the probability of the inverse $\beta$ transformation must be very low. However, with the availability of very high antineutrino flux in nuclear reactors, it has been possible to carry out such an experiment to demonstrate the interaction between an antineutrino and a nucleus, thereby providing direct evidence for the existence of the neutrino.

The experiment was first carried out by F. Reines and C.L. Cowan, Jr in 1956 in the U.S.A. Schematic diagram of the experimental arrangement is shown in Fig. 5.38. The intense antineutrino beam from a high flux nuclear reactor enters into two tanks of water in which the salt $\text{CdCl}_2$ is dissolved. There were three large detector tanks, containing a scintillating liquid separated by the target tanks containing the $\text{CdCl}_2$ solution. The scintillations in the detector tanks could be observed through 110 photomultiplier tubes. The entire set up was surrounded by
lead-paraffin cover and placed deep underground to protect it from neutrons and γ-radiation.

The interaction between an antineutrino and proton in the target tank according to Eq. (5.19-2) produced a neutron and a positron. Soon after its formation, the positron is annihilated which resulted in the emission of two annihilation photons, each of energy 0.511 MeV. These two photons produced scintillations within the scintillating liquid resulting in two electrical pulses within two photomultiplier tubes, the outputs of which were fed to a coincidence circuit. The neutron produced in the inverse beta transformation suffered repeated collisions with the protons in the water tank and was slowed down. It was ultimately absorbed by a cadmium nucleus, which has a high thermal neutron absorption cross-section (see Ch.XIII) resulting in the emission of a few γ-quanta with a total energy of 10−3 MeV. These γ-quanta were also registered by the detector liquids.

Signals from the detectors were led through coaxial cables to the input of a special electronic device, which could analyze their amplitudes and coincidence displacement time. The signals could also be photographed by a triple-beam oscillograph (see Fig. 5.39). There was time-delay ranging from 1 to 25 μs between the two γ-pulses.

In their experiment, Reines and Cowan recorded 2.88 double pulses per hour like those shown in Fig. 5.39 from observations lasting for about 1400 hours. From their observations, they concluded that the cross-section of interaction between a ν and a nucleus was σ ~ 10−45 m² or 10−19 barn.

This value is in agreement with the two-component theory of the neutrino.

Soon afterwards, B. Davis experimentally demonstrated the impossibility of process (5.19-3) to transform a neutron into a proton by antineutrino absorption through non-occurrence of the following reaction:

\[ ^{37}\text{Cl} + \nu \rightarrow ^{37}\text{Ar} + \beta^+ \]

Reines and Cowan's experiment can be considered as the first direct confirmation of the existence of the neutrino. The experiment of Davis, on the other hand, confirmed the conclusions of double β-decay study, that neutrino and antineutrino were different.

**5.20 Forms of the β-decay interaction**

All the particles involved in β-decay are fermions. So each of them must be represented by a 4-component Dirac spinor. The Fermi theory of β-decay considers a vector form of the weak interaction which involves an operator similar to the one used in the theory of electromagnetic transitions. However, this theory is unable to explain the data on all allowed β-transitions. In particular, it cannot explain the β-decay of ⁶He. Thus arose the necessity of modifying Fermi's theory.

As seen in § 5.7 the β-decay interaction can be constructed from the linear combination of the wave functions of the two nucleons and the two leptons (see Eq. 5.7-6). Since each of these is a four component Dirac spinor, it is possible to construct 4³ or 256 linearly independent forms of interactions. However, the number is drastically reduced by the imposition of Lorentz invariance. The five covariant types, which are finally obtained are the following:

- Scalar
  \[ S, \Delta I = 0, \text{ No.} \]
- Vector (Polar)
  \[ V, \Delta I = 0, \text{ No.} \]
- Tensor (anti-symmetric)
  \[ T, \Delta I = 0, \pm 1, \text{ No.} \]
- Axial vector
  \[ A, \Delta I = 0, \pm 1, \text{ No.} \]
- Pseudo scalar
  \[ P, \Delta I = 0, \text{ Yes.} \]

Each of these interactions has a definite form of the perturbation operator \( H_i' \) which conserves parity and angular momentum in weak interaction. In the general case, we can write \( H'_i = \sum C_i H_i' \) where \( i \) takes on the five values corresponding to the five possible forms of the interaction. The total number of parameters is still too large (10) because the coefficients \( C_i \) are complex. However, by making some simplifying assumptions regarding the forms of the interactions, the number of parameters can be further reduced. Thus in view of the very weak interaction of the leptons with the nucleons and neglecting Coulomb effect, it is possible to write the lepton wave functions as plane waves, as was done in § 5.7. The theoretical predictions for allowed transitions which follow have been verified for specific cases, which have been discussed in detail earlier.

Each of the five interactions gives rise to its own selection rules. Thus the scalar (S) and vector (V) versions give rise to the Fermi selection rules, while the Gamow-Teller selection rules follow from the tensor (T) and the axial vector (A) versions. The pseudo-scalar (P) version however is not suitable for describing allowed transitions, since it predicts parity change with \( \Delta I = 0 \).

As seen above, for parity conserving interactions, there are five complex coefficient \( C_i \) giving ten real coefficients. However, assuming time-reversal invariance of the weak interaction (see Ch. XVIII), the coefficients \( C_i \) become real, so that we are left with only five coefficients.

For allowed transition \( C_p = 0 \) for the \( P \)-interaction. So we finally have the four coefficients \( C_S, C_V, C_T \) and \( C_p \) of which only one \( (C_S \text{ or } C_V) \) is predominant at a time in the Fermi type transition. Similarly for the G-T type, only one of the two \( (C_T \text{ or } C_p) \) predominates at a time. Thus we can have any one of the following four combinations: \( ST, SA, VT, \text{ or } VA \). An analysis of the experimental data reduces the choice to finally \( ST \text{ or } VA \). However, the number of coefficients still remains four.
The experimental results on the shape of the forbidden \(\beta\)-spectrum, as also on the \(\beta-V\) and \(\beta-\gamma\) angular correlations finally reduce the choice further to vector interaction (V) for the Fermi and to axial vector interaction (A) for the G-T transitions. Thus the number of parameters we get finally is only two \((C_V \text{ and } C_A)\).

We have already discussed about the experimentally determined values of the strengths \(g_F\) and \(g_{GT}\) of the interactions in the two cases.

**V - A interaction:**

The results of the \(^{152}\text{Eu}\) experiments by Goldhaber and others (see § 5.17) have definitely established that the helicity of the neutrino is negative as in a left-handed screw. The results of the longitudinal polarization experiments of the charged particles in beta-decay along with the above results have indicated that the Gamow-Teller interaction is vector axial (A) in character. This is consistent with the evidence from the electron neutrino correlation experiments. On the other hand, all parity experiments point to the conclusions that the Fermi type transitions are of the vector type (V) and the antineutrino has positive helicity as in a right-handed screw. In both cases the helicities of the massless leptons (v and \(\bar{v}\)) are \(\pm 1\) and \(\pm \omega\) for the electrons and positrons.

The \(\beta\) values of both Fermi type and G-T type transitions for which the nuclear matrix elements known show that the ratio of the coupling constants \(C_V/C_A = V_{GR}/C_A = 1.23\) (see § 5.9). Experiments with polarized neutrons also show that the sign of this ratio is negative. Because of this the beta-decay interaction is said to be \(V-A\) (V minus A) interaction.

### References